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EVEN ALTERNATING SPLINE QUADRATURE FORMULAS

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1. Introduction. In a recent paper Busenberg and Fisher [1] developed odd degree spline quadrature formulas for sufficient smooth functions satisfying Lidstone boundary conditions. Using the method of *interpolation by translation* we will establish a systematic approach to the *even alternating spline quadrature* which yields beside the known formulas new even degree spline quadrature formulas.

2. Interpolation by translation. For the reader's convenience we briefly recall the method of interpolation by translation (cf.[2]).

Suppose that $g:\mathbb{R} \to \mathbb{R}$ is a 2π-periodic continuous function with an absolute and uniform convergent Fourier series and $n \in \mathbb{N}$.
We consider the uniform mesh $t_j := 2\pi j/n$, $j \in \mathbb{Z}$, and the linear space

$$V_n^1(g) := \text{span}\,\{ 1,\ g(t-t_j)-g(t) \mid j=1,\dots,n-1 \} \tag{2.1}$$

of the modified translates of g .

The basis functions

$$g_k(t) := \sum_{j=0}^{n-1} g(t-t_j)\exp(ikt_j)\ , \quad k=1,\dots,n-1 . \tag{2.2}$$

are elements of $V_n^1(g)$ satisfying

$$g_k(t_j) = g_k(0)\exp(ikt_j)\ , \quad 1\le k\le n-1;\ j \in \mathbb{Z} .$$

We assume that the discrete Fourier coefficients $c_{k,n}(g) = g_k(0)/n$ of g do not vanish for $k=1,\dots,n-1$ and define the function

$$L_n(t) := \frac{1}{n}\,[1 + \sum_{k=1}^{n-1} g_k(t)/g_k(0)\ ] \in V_n^1(g) . \tag{2.3}$$

It enjoys the interpolation conditions $L_n(t_j) := \delta_{0,j}$, $0\le j<n$ and we have

<u>Theorem 1</u> *Suppose that* $f \in C_{2\pi}$. *Then* $S_n(f)(t) := \sum_{k=0}^{n-1} f(t_k)\, L_n(t-t_k)$ *is the unique function in* $V_n^1(g)$ *that interpolates* f *at the nodes* $t_j=2\pi j/n$, $j \in \mathbb{Z}$.

An important example is connected with the *Bernoulli* functions

$$P_q(t) = \sum_{k\neq 0} (ik)^{-q} \exp(ikt) .$$

It was proved by Meinardus [4] that $\{1, P_q(\cdot-t_1)-P_q, \ldots, P_q(\cdot-t_{n-1})-P_q\}$ form a basis of the space $S(q-1,n)$ of *periodic polynomial splines* of degree q-1 with spline knots t_j, $j \in \mathbb{Z}$. For $n:=2m+1$, $m \in \mathbb{N}$ theorem 1 is applicable to $V_n^1(P_q) = S(q-1,n)$.

<u>3. Interpolation of odd periodic functions.</u> In this section let $n=2m$, $m \in \mathbb{N}$, and g be even. Then the basis functions g_k, $k=1,\ldots,2m-1$, are in general complex valued with

$$\operatorname{Re} g_k(t) = g(t) + g(t-\pi)\cos(k\pi) + \sum_{j=1}^{m-1} \cos(kt_j)\,[g(t-t_j) + g(t+t_j)] , \qquad (3.1)$$

$$\operatorname{Im} g_k(t) = \sum_{j=1}^{m-1} \sin(kt_j)\,[g(t-t_j) - g(t+t_j)] .$$

Obviously, $\operatorname{Re} g_k$ is even and $\operatorname{Im} g_k$ odd for $k=1...,n-1$.

We define the function

$$N(t) := \frac{1}{n} \sum_{k=1}^{m-1} [g_k(t)/g_k(0) + g_{n-k}(t)/g_{n-k}(0)]. \qquad (3.2)$$

Since $g_{n-k}(t) = \overline{g_k(t)}$, $k=1,\ldots,m-1$; $t \in \mathbb{R}$, one gets

$$N(t) = \frac{1}{m} \sum_{k=1}^{m-1} \operatorname{Re} g_k(t)/\, g_k(0) . \qquad (3.3)$$

Thus, $N(t)$ is even again.

It follows that the odd functions

$$N_k(t) := N(t_k-t)-N(t_k+t) \in V_n^1(g) , \quad 1\le k\le m-1,$$

satisfy the interpolation properties $N_k(t_j) = \delta_{k,j}$, $1\le k,j\le m-1$ (cf.[2]). As a consequence we obtain

Theorem 2 *Let* $f \in C_{2\pi}$ *be odd. Then the odd function*

$R_m(f)(t) := \sum_{k=1}^{m-1} f(t_k)\ [N(t_k - t) - N(t_k + t)]$ *is the unique function in* $V_n^1(g)$ *that interpolates* f *at nodes* $t_j = \pi j/m$, $j \in Z$.

Remark: Theorem 2 is applicable to $V_n^1((-1)^r P_{2r}) = S(2r-1,n)$ which is the space of periodic polynomial splines of degree $2r-1$.

4. Shifted generating functions. We make again the assumption that $n=2m$, $m \in \mathbb{N}$. The following result was proved in [2].

Proposition 1 *Suppose that the odd function* $g(t) := \sum_{k>0} d_k \sin(kt)$ *satisfies* $d_k > d_{k+1} > 0$, $k \in \mathbb{N}$. *Furthermore, let* $G(t) := g(t-\pi/n)$. *Then* $G_k(0) = \sum_{j=1}^{n-1} G(-t_j) \exp(ikt_j) \neq 0$, $k=1,\dots,n-1$ *and theorem 1 is applicable to the shifted function* G *of* g .

First we introduce the shifted uniform mesh $z_j := t_j + \pi/n$, $j \in Z$.
The basis functions G_k in $V_n^1(G)$ can be expressed by

$$G_k(t) = \exp(-ikz_0)\ g_k^*(t) \tag{4.1}$$

where

$$g_k^*(t) := \sum_{j=0}^{n-1} g(t-z_j) \exp(ikz_j)\ , \quad k=1,\dots,n-1. \tag{4.2}$$

Note that

$$\operatorname{Re} g_k^*(t) = \sum_{j=0}^{m-1} \cos(kz_j)\ [g(t-z_j) + g(t+z_j)]$$

and

$$\operatorname{Im} g_k^*(t) = \sum_{j=0}^{m-1} \sin(kz_j)\ [g(t-z_j) - g(t+z_j)]\ . \tag{4.3}$$

Since g is odd $\operatorname{Im} g_k^*$ is even and $\operatorname{Re} g_k^*$ odd for $k=1,\dots,n-1$.

Analogously to (3.2) we define the function

$$N(t) := \frac{1}{n} \sum_{j=0}^{m-1} [G_k(t)/G_k(0) + G_{n-k}(t)/G_{n-k}(0)]. \tag{4.4}$$

Then $N \in V_n^1(G)$ can be computed to

$$N(t) = \frac{1}{m} \sum_{j=0}^{m-1} \operatorname{Re}\ [G_k(t)/G_k(0)]$$

$$= \frac{1}{m} \sum_{j=0}^{m-1} \operatorname{Re}\ [g_k^*(t)/g_k^*(0)]\ ,\ \text{i.e.}$$

$$N(t) = \frac{1}{m} \sum_{k=1}^{m-1} \text{Im } g_k^*(t)/\text{Im } g_k^*(0) . \quad (4.5)$$

Since $N(t)$ is again even the odd functions

$$N_k(t) := N(t_k - t) - N(t_k + t), \quad 1 \le k \le m-1,$$

are elements of $V_n^1(G)$ satisfying $N_k(t_j) = \delta_{k,j}$, $1 \le k, j \le m-1$.

Theorem 3 *Let* $f \in C_{2\pi}$ *be odd. Then the odd function* $R_m(f)(t) := \sum_{k=1}^{m-1} f(t_k) \; [N(t_k - t) - N(t_k + t)]$ *is the unique function in* $V_n^1(G)$ *that interpolates* f *at the nodes* $t_j = \pi j/m, \; j \in Z$.

Remark: Theorem 3 is applicable to $V_n^1((-1)^r P_{2r+1}(t - \pi/n))$ which is the space of *periodic midpoint splines* of degree $2r$ with spline knots $(2j+1)\pi/n, \; j \in Z$ ([2,5]).

5. Even alternating quadrature formulas.

Theorem 2 and Theorem 3 are suited for the numerical integration of odd periodic functions. For smooth generating functions g respectively $G = g(\cdot - \pi/n)$ the odd interpolants $R_m(f)$ satisfy even alternating boundary conditions, i.e. they have vanishing even order derivatives at the points 0 and π. For this reason the interpolatory quadrature formula based on the operator R_m is called *even alternating quadrature formula*:

$$\int_0^\pi f(t)dt = \int_0^\pi R_m(f)(t)dt + E_m(f) = \sum_{k=1}^{m-1} a_k f(t_k) + E_m(f) \quad (5.1)$$

where E_m is the error functional ([1,3]).

Our objective is to compute the weights a_k, $k=1,\dots,m-1$.

Suppose that $h \in C_{2\pi}^1$ is an (odd or even) primitive of the (odd or even) generating function g , i.e. $h'(t) = g(t)$.

If g is an even function with non-vanishing discrete Fourier coefficients we define the function $Q(t)$ by

$$Q(t) := \frac{1}{n} \sum_{k=1}^{m-1} [h_k(t)/g_k(0) + h_{n-k}(t)/g_{n-k}(0)] . \quad (5.2)$$

Analogously to the computations done in section 3 we get

$$Q(t) = \frac{1}{m} \sum_{k=1}^{m-1} \text{Re } h_k(t)/g_k(0). \quad (5.3)$$

Thus, $Q'(t) = N(t) \in V_n^1(g)$ and $Q(-t) = -Q(t)$.

If g is an odd function of that type we considered in section 4 and G the connected shifted function we define

$$Q(t) := \frac{1}{n} \sum_{k=1}^{m-1} [h_k^*(t)/g_k^*(0) + h_{n-k}^*(t)/g_{n-k}^*(0)] . \tag{5.4}$$

Then one obtains the representation

$$Q(t) = \frac{1}{m} \sum_{k=1}^{m-1} \text{Im } h_k^*(t)/\text{Im } g_k^*(0). \tag{5.5}$$

Obviously , $Q(t)$ is again an odd primitive of $N(t) \in V_n^1(G)$ defined in (4.4).

<u>Theorem 4</u> *The weights of the even alternating quadrature formula are given by* $a_k = 2\ [Q(t_k) + Q(t_{m-k})]$, $k=1,\ldots,m-1$.

<u>Proof:</u> The construction of the interpolation functions $R_m(f)$ implies

$$a_k = \int_0^{\pi} [N(t_k-t) - N(t_k+t)]\, dt = 2\ [Q(t_k) - Q(t_{k-m})]$$
$$= 2\ [Q(t_k) + Q(t_{m-k})].$$

<u>Remark:</u> It immediately follows from theorem 4 that
$a_k = a_{m-k}$, $k=1,\ldots,m-1$.

The weights can be computed only if besides g the function h is still known. This is the case for the even alternating spline quadrature formulas. Here we have $g = P_q$ and $h = P_{q+1}$.

For q=2r we have *splines of odd degree* 2r-1. The corresponding weights were first given by Busenberg and Fisher [1] in terms of the zeros of the Euler-Frobenius polynomials.

If q=2r+1 we obtain a new class of even alternating spline quadrature formulas based on *midpoint splines of even degree* 2r.

d	a_1	a_2	a_3	a_4
2	0.4376178675	0.3849857548	0.3940602570	0.3922453566
4	0.4517242267	0.3718116415	0.4010756044	0.3873448812
6	0.4573724015	0.3637685604	0.4088859476	0.3799731633
8	0.4603491999	0.3587402360	0.4148626841	0.3737623836

Table 1: Weights of the even alternating quadrature formula for midpoint splines of even degree d (m = 8)

We finish this section by applying the even alternating spline quadrature formula of degree d to the even function f(t) = sin(t)exp(cos(t)). The exact value of the integral is 2.350402387... .

d	m=2	m=4.	m=8	m=16
1	0.7796060605	0.1648373328	0.0400060298	0.0099361908
2	0.4305402101	0.0309133095	0.0013836709	0.0000764155
3	0.3869069788	0.0209149606	0.0005257246	0.0000234671
4	0.3607270401	0.0129678888	0.0000967891	0.0000010469
5	0.3541820554	0.0092069569	0.0000342317	0.0000002213
6	0.3515983752	0:0067242564	0.0000115957	0.0000000196
7	0.3508133132	0.0052456167	0.0000051811	0.0000000038
8	0.3505369907	0.0043326133	0.0000026213	0.0000000006

Table 2: Error of numerical integration of f(t) = sin(t) exp(cos(t))

References

1 S.N. BUSENBERG and FISHER: Spline quadrature formulas. Journal of Approximation Theory 42 (1984) , 212-238.

2 F.-J. DELVOS: Interpolation of odd periodic functions on uniform meshes. ISNM 74 (1985) , 105-121.

3 F.-J. DELVOS: Numerical integration of odd periodic functions. ZAMM (1987) (to appear) .

4 G.MEINARDUS: Periodishe Splinefunktionen. In "Spline functions, Karlsruhe 1975" (K.Böhmer, G.Meinardus, W.Schempp, eds.). LNM 501 (1976) , 177-199 .

5 Y.N. SUBBOTIN: On relations between finite differences and the corresponding derivatives. Proceedings of the Steklov Institute of Mathematics 78 (1965), 23-42.

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