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STABILITY RESULTS FOR A CLASS OF NONLINEAR DIFFERENTIAL INCLUSIONS IN BANACH SPACES

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1. Introduction. We consider differential inclusions in nonseparable Banach spaces with dissipative type conditions on the right-hand side. Such type of differential inclusions is considered in [5]. Where the existence of solutions has been proved, adapting the Euler-Caushy method, presented in [1]. In this work the Kamke function $g: R \times R^+ \longrightarrow R^+$ is used, where $g(\cdot,s)$ is measurable, $g(t,\cdot)$ - continuous. The differential equation $\dot{s}(t)=g(t,s(t))$ has a solution on the whole interval $[t_0,t_1]$ for every initial condition $s(t_0)=s_0$, $g(t,0)=0$, and $s(t)\equiv 0$ is the unique solution of the equation with the initial condition $s(t_0)=0$

Our results are new even in R^n. We prove a theorem corresponding to theorem 2.4.1. in [1].

Theorem ([1]). Assume that $R_0 := [a,b] \times E$

(i) $f \in C[R_0,E]$, $|f(t,x)| \leq M$ on R_0 and $\alpha = \min(a, b/(M+1))$

(ii) $\min_{v \in J(x-y)} \langle v, f(t,x)-f(t,y)\rangle \leq g(t,|x-y|).|x-y|$

where g is the Kamke function. Then:

1) There exists an unique solution of $\dot{x}=f(t,x)$ for each initial condition x_0 on $[t_0,t_1]$

2) This solution depends continuously on f and x_0.

2. Notations. Let B and B^* be a Banach space and its conjugate. We denote by $cl_b B$ and by $clco_b B$ the set of all closed bounded subsets of B and the set of all convex sets in $cl_b B$. The duality product between B and B^* will be denoted as $\langle\cdot,\cdot\rangle$. By $\delta(\cdot,U)$ we mean the support function of the set U. For the set C denote by $cl\,C$ and $co\,C$ the closed, respectively the convex hull of C. The function $h(A,B)$ is the Hausdorff distance between the closed sets A and B. Further $j: B \to clco_b B^*$ is the duality mapping, i. e. for each $x \in B$, $y \in j(x)$ $\langle y, x\rangle = |x|^2$, where $|x|$ is the norm of x. When $a \in B$, $A \in cl_b B$ then $d(a,A) := \inf_{b\in A} |b-a|$. The interval $I := [0,T]$ is closed in R. Through $C(I,B)$ denote the space of all continuous functions on I with values in B, provided with the norm $\|x(\cdot)\| := \max_{t\in I} |x(t)|$. Through $L_1[I,B]$ denote the space of all Bochner integrable functions from I to B, equipped with the norm $|x(\cdot)|_{L_1} := \int_I |x(\tau)|\, d\tau$

3. The system description and the main result. Here we consider the system:

(1) $\dot{x}(t) \in F(t,x(t))$, $x(0)=x_0$, $t \in I$, under the assumptions:

A1. The function $F(\cdot,\cdot)$ from $I\times B$ to $cl_b B$ is strongly measurable in t for each x and satisfies Scorza-Dragony condition on the bounded subsets of $I\times B$. ([3])

A2. The space B is such that j is upper semicontinuous with respect to the strong topologies of B and B^*.

A3. If x, y belong to B and $\hat{x} \in F(t,x)$, then for every $\varepsilon > 0$ and $J \in j(x-y)$ there exists $\hat{y}_\varepsilon \in F(t,y)$ such that:

a) $\langle J, \hat{x} - \hat{y}_\varepsilon\rangle \le g(t,|x-y|)\,|x-y| + \varepsilon$

b) $|\hat{x} - \hat{y}_\varepsilon| \le \varepsilon + f(|x-y|)$, where $f: R^+ \to R^+$ is continuous and $f(0)=0$.

Now we present our main result:

Theorem 1. Let $\{F_n(t,x)\}_{n=1}^{\infty}$ be a sequence of maps with closed values in B, which satisfy A1-A3 (with the same g) and $h(F_n(t,x), F(t,x)) \xrightarrow[n\to+\infty]{} 0$ uniformly on each closed bounded subset of $I \times B$. Then:

i) Each solution $y(\cdot)$ of (1) is an uniform limit of the corresponding solutions of

$$(2) \quad \dot{x}(t) \in F_n(t,x(t)), \quad x(0)=x_0.$$

ii) For every uniformly converging sequence $\{y_n(\cdot)\}_{n=1}^{\infty}$ of the corresponding solutions of (2) with limit $y(\cdot)$ there exists an uniformly convergent sequence of solutions of (1) with the same limit.

Further the set of solutions of (1) depends continuously on the initial condition x_0. ■

4. Proof of the main result.

We essentialy use theorem 2 from [2], which says:

Theorem ([2]). Let F and G mapping I into $cl_b B$ be strongly measurable, and $f(\cdot)$ be strongly measurable selection of $F(\cdot)$. Then for each positive ε and each strongly measurable essentially bounded function $S: I \to B^*$ there exists a strongly measurable selection $g(\cdot)$ of $G(\cdot)$ such that:

$$\langle S(t), f(t)-g(t)\rangle < \delta(S(t), F(t)) - \delta(S(t), G(t)) + \varepsilon;$$

One can choose g such that:

$$\langle S(t), f(t)-g(t)\rangle > \delta(S(t), F(t)) - \delta(S(t), G(t)) - \varepsilon;$$

Definition 1. Let $\varepsilon>0$. The function $y(\cdot)$ is said to be ε-solution of (1) if $y(\cdot)$ is differentiable a. e. in I and satisfies the following conditions:

i) $y(t)=x_0+\int_0^t \dot{y}(\tau)d\tau$; here $\dot{y}$ denotes the derivative of y.

ii) $d(\dot{y}(t), F(t,y(t))<f_\varepsilon$ a. e. in I, here f_ε is a positive func-

tion with L_1 norm less than ε.

The Euler brocken are defined only for continuous functions. We shall give a proper definition also in case $F(\cdot,x)$ is strongly measurable. Let $\{0=\tau_0<\tau_1<\ldots<\tau_n=T\}$ be subdivision of I.

Definition 2. Euler brocken is called any function $z(t)$ such that $z(t)\equiv z(\tau_i)+\int_{\tau_i}^{t} f(s)ds$. Here $f(\cdot)$ is strongly measurable selection of $F(t,z(\tau_i))$ for $t\in[\tau_i,\tau_{i+1})$, which existence is proved in [2]. We determine

$z(\tau_{i+1})$ as $\lim_{t\to\tau_{i+1}} z(t)$, $z(0)=x_0$

Following [1] and [5] one can prove that under the assumptions A1-A3 for each $\varepsilon>0$ there exists an ε-solution and that this ε-solution is extendable on the whole interval I.

Lemma 1. Let ε_1, ε_2 be positive numbers and $x(\cdot)$ be ε_1-solution of (1). Then there exists ε_2-solution $y(\cdot)$ of (1) such that:

$|x(t)-y(t)|\le v(t)$, where

$$\dot v(t)= g(t,|x(t)-y(t)|)+\varepsilon_1+2\varepsilon_2+\varepsilon_2^2/2,\quad v(0)=0.$$

Proof: It is known (see [6]) that the left derivative of the norm $d_-|x(t)-y(t)|$ exists a. e. in I and:

$d_-|x(t)-y(t)|\le 1/|J(t)|\cdot\langle J(t),\dot x(t)-\dot y(t)\rangle$, where

$J(t)\in j(x(t)-y(t))$. Using theorem 2 from [2] and assumptions A2 and A3, we can construct $\dot y(t)$ such that $y(\cdot)$ satisfies the conditions of the lemma. More precisely: let $\{\tau_1,\ldots,\tau_{n_1}\}$ be the partition of I corresponding to x. We set $\dot y(t)\equiv\dot x(t)$ on $[0,\chi_1]$, where χ_1 is the maximal number such that $y(t)$ is an ε_2-solution $(\varepsilon_2>\varepsilon_1)$. In accordance with theorem 2 in [2] and A3 we choose $\dot y(\cdot)$ as a selection of $F(t,y(\chi_1))$ such that:

$$d_-|x(t)-y(t)|\le 1/|J(\chi_1)|\cdot\langle J(\chi_1),\dot x(t)-\dot y(t)\rangle + d(\dot x,F(t,x))+$$
$$+d(\dot y,F(t,y))+\varepsilon_2^2/2<$$
$$<1/|J(\chi_1)|\cdot\langle J(\chi_1),\dot x(t)-\dot y(t)\rangle+\varepsilon_1+\varepsilon_2+\varepsilon_2^2/2,$$

(because $d(\dot{x}, F(t,x)) < \varepsilon_1$ and $d(\dot{y}, F(t,y)) < \varepsilon_2$). Since $j(\cdot)$ is upper semicontinuous, there exists χ such that the conditions of lemma 1 are fulfilled also on $[\chi_1, \chi_2)$. Proceeding in the same manner we determine χ_3 such that the lemma is true also on $[\chi_2, \chi_3)$ and so on. We shall prove that $y(\cdot)$ is extendable on the whole interval I. Suppose the contrary, that is $\chi_1 < \chi_2 < \ldots < \chi_n < \cdots < \chi < T$. Using the continuity of $x(\cdot)$, $y(\cdot)$, $F(t, \cdot)$ and the usc of $j(\cdot)$ one can continue aw in [1] p. 39, and obtain contradiction. Then finally:

$|x(t) - y(t)| \leq r(t)$, where $\dot{r}(t) = g(t, |x(t) - y(t)|) + \varepsilon_1 + 2\varepsilon_2 + \varepsilon_2^2/2$ and $r(0) = 0$. The proof is complete. ∎

<u>Proof of theorem 1:</u> Let $x(\cdot)$ be a solution of (1) . From the conditions of the theorem it follows that there exists a neighbourhood $\mathcal{U}$ of x such that $\lim(F_i(t,z), F(t,z)) = 0$ uniformly in $I \times \mathcal{U}$. Hence for every $\varepsilon > 0$ there exists $n(\varepsilon)$ such that $x_n(\cdot)$ is ε-solution of (2) when $n \geq n(\varepsilon)$. The function $F_i(\cdot,\cdot)$ satisfies the assumptions A1-A3, where the function $g(\cdot,\cdot)$ is the same as one for $F(\cdot,\cdot)$. Then since g is continuous with respect to the second argument, following [5] we obtain that for each fixed $\varepsilon > 0$ it is possible to determine a sequence of ε_n-solutions $x_n(\cdot)$ of (2) (where $\varepsilon_1 > \varepsilon_2 > \cdots > \varepsilon_n > \ldots \to 0$) such that:

$\|x(\cdot) - x_n(\cdot)\| \leq \sum_{i=1}^{\infty} r_i(t)$. Here

$r_i(0) = 0 \quad \dot{r}_i(t) = g(t, r_i(t)) + \varepsilon_i + \varepsilon_{i+1} + \varepsilon_i^2/2$ (we set $\varepsilon_1 = \varepsilon$). The difficulty is to choose $\{x_i(\cdot)\}_{i=1}^{\infty}$, so that it would be a Cauchy sequence. Since $g(t, \cdot)$ is continuous, after determining $\dot{x}_i(\cdot)$ one can choose $\dot{x}_{i+1}(\cdot)$ on each interval: $[\chi_j^{(i+1)}, \chi_{j+1}^{(i+1)})$ as ε_{i+1}^2 projection of $x_i(\cdot)$ on the set $F(t, x_{i+1}(\chi_j))$. Such a choice is possible thanks to proposition 1 in [2]. Here $\{\chi_j^{(i)}\}_{j=1}^{\infty}$ is the corresponding to x_i partition of the interval I. Now using assumption A3 one can prove that:

$\|x_i(t) - x_{i+1}(t)\| \le r_i(t) + \varepsilon_{i+1}^2/2$.

Moreover following [5] one can prove that $\{\dot{x}_i(\cdot)\}_{i=1}^{\infty}$ is indeed a Cauchy sequence in $L_1(I,B)$. It follows from the properties of g (see [1]) that the first part is proved. The proof of the seqond part is similar. ∎

Remark. If we use the generalised Euler brocken and assume that $F(\cdot,\cdot)$ is continuous we can prove such a result for arbitrary Banach space B(without assumption A2). The generalised Euler brocken are constructee as the standart Euler brocken, but we permit points of density for the subdivision $\{\tau_i\}_{i=1}^{\infty}$. Assuming that F has convex and (strongly) compact values one can prove rheorem 1 when A3 is relaxed by:

A3' For every x, y from B and each $J \in j(x-y)$

$$\delta(J, F(t,x)) - \delta(J, F(t,y)) \le g(t, |x-y|)\,|x-y|,$$

(see [7]).

Example. Let c_0 be the space of all bounded sequences such that $\lim_{i\to\infty}|x_i| = 0$, provided with the norm $|x| := \max_i |x_i|$. Then the following system satisfies all the assumptions of theorem 1:

$$x_n \ge \operatorname{sign}(-x_n)\cdot|x_n|^{1/2}\cdot|x_{n+1}|^{1/3} + 1/n - (x_n)^{1/3},$$
$$x_n \le \operatorname{sign}(-x_n)\cdot|x_n|^{1/2}\cdot|x_{n+1}|^{1/3} + 1/n + |x_n| + 1, \qquad n = 1 \div \infty.$$

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