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AN AVERAGED MODULUS OF CONTINUITY FOR MULTIVALUED MAPS AND ITS APPLICATIONS TO DIFFERENTIAL INCLUSIONS *

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1. Introduction. In the paper we present an error estimate for a discrete approximation applied to the differential inclusion

(1) $\dot{x} \in F(x,t)$, $t \in [0,1]$, $x(0) = x^0$.

Let $\Gamma^N = \{ t_i = ih , i=0,1,\dots,N \}$ be a uniform grid over $[0,1]$, $h = 1/N$. Using the simplest Euler scheme we get the approximation

(2) $x_{i+1} \in x_i + h F(x_i, t_i)$, $i = 0,1,\dots,N-1$, $x_0 = x^0$.

Denote by T_N the set of all vectors $x^N = \{ x(t_i), i=0,\dots,N \}$ where $x(\cdot)$ is a solution of (1) and let D_N be the set of vectors $y^N = \{ x_i, i=0,\dots,N \}$ that satisfy (2). Our main result follows :

Theorem. Let the following conditions hold:

(i) The map $F: R^n x[0,1] \to 2^{R^n}$ is convex and compact valued, continuous in $x \in R^n$ for fixed $t \in [0,1]$ and measurable in $t \in [0,1]$ for fixed $x \in R^n$.

(ii) There exists a constant $k>0$ and a Riemann integrable $a:[0,1] \to R^n$, such that for every $(x,t) \in R^n x[0,1]$ $|F(x,t)| \le k|x| + a(t)$.

(iii) For every bounded set $X \subset R^n$ there exists a constant $L(X)$ such that for any $t \in [0,1]$ and $x, y \in X$ $|F(x,t) - F(y,t)| \le L(X)|x-y|$.

Let $H(T_N, D_N)$ be the Hausdorff distance between the sets T_N and D_N. Then there exists a constant c independent of N such that

(3) $H(T_N, D_N) \le c (\tau(F,h) + h)$,

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where τ is the modulus of continuity of F defined in Section 2. Here we mention two important properties of τ presented in Section 2.

(A) Let V_1 be a bounded set containing all trajectories of (1) (it exists by (ii)). Suppose that F(x, t) is continuous function of x uniformly in t and continuous in t for all $x \in V_2$ and almost all t in [0, 1] . Then $\tau(F, h) \to 0$ as $h \to 0$.

(B) Suppose that there exists a constant W > 0 such that

$$\sup \{ \sum_{i=1}^{k-1} \sup[H(F(x, t_{i+1}), F(x, t_i))^p, x \in V_1], k \in N, 0 \le t_1 \le \ldots \le t_k \le 1 \} \le W$$

i. e. $F(x, \cdot)$ has a bounded variation in t uniformly in x. Then there exists a constant $c_1 > 0$ such that $\tau(F, h) \le c_1 h$.

To the autors' knowledge there are only few results in the numerical analysis of differential inclusions. Consider the solutions of (2) as piecewise linear functions across the grid points. Then, on the conditions (i)-(ii) using Theorem 3.1.7. from Clarke [5], p. 118 one can prove the following: any sequence of discrete solutions $\{x^N\}$ has an accumulation point in $C(R^n, [0, 1])$ and every accumulation point is a solution of (1). Taubert and Ansorge [1, 2] obtained analogous results for a general class of multistep methods. We should note that, compared with the Euler scheme, the analysis of higher order integration schemes is considerably more complicated.

Supposing that the trajectories of (1) are in a bounded set $D \subset R^n$ and that F is compact and convex valued and Lipschitz continuous on $D \times [0, 1]$, Pshenichnij [6] proved the following : for any solution x of (1) there exists discrete trajectory x^N of (2) and a constant C>0 such that $\max_i \| x_i^N - x(t_i) \| \le C h$ for sufficiently small h.

We use stronge assumption (ii), however, our theorem :
1) establishes an error estimate between the sets of solutions ,
2) relaxes the continuity requirements for $F(x, \cdot)$.

We mention also the independent papers by Dontchev [3] and Mordukhovich [4] concerned with discrete approximations to minimum problems involving differential inclusions. These papers give

conditions for convergence of the optimal value of the discretized problem to the value of the continuous one.
techniques of the averaged moduli of smoothness, recently developed for single-valued maps by Sendov et al., see [7] and applied to differential equation in Andreev et al. [8].

2. The modulus of continuity. Consider a set-valued map F defined over the interval [0, 1] with compact images in R^n. Let $t_o \in [0,1]$ and $h > 0$ be fixed.

Define the local modulus of continuity

$$\omega(F, to, h) = \sup\{ H(F(t'), F(t")),\ t', t" \in [t_o - h/2, t_o + h/2] \cap [0,1] \},$$

where H denotes the Hausdorff distance. We define the L_p-averaged modulus of continuity for the map F :

$$(4) \quad \tau(F,h)_p = \| \omega(F, \cdot, h) \|_{L_p[0,1]}$$

Denote $\tau(F,h) = \tau(F,h)_1$.

In the sequel we use the p-variation of the map F, defined for $p \in [1, +\infty)$ as

$$V_p(F) = \sup \{ \sum_{i=1}^{K-1} [H(F(t_{i+1}), F(t_i))]^p,\ K \in N,\ 0 \le t_1 \le \ldots \le t_K \le 1 \}$$

Some important properties of the modulus $\tau(F,h)_p$ are given below:

1^o. For $h' \le h"$ $\tau(F,h')_p \le \tau(F,h")_p$;

2^o. $\tau(F_1 + F_2, h)_p \le \tau(F_1,h)_p + \tau(F_2,h)_p$, where $F_1 + F_2$ is the algebraic sum of F_1 and F_2 ;

3^o. $\tau(F, k\delta)_p \le 2k^2 \tau(F,\delta)_p$ for any positive integer k ;

4^o. If F is of bounded p-variation, then $\tau(F,h)_p \le (V_p(f)h)^{1/p}$;

5^o. $\lim_{h \to 0} \tau(F,h)_p = 0$ iff F is Hausdorff continuous for almost all $t \in (0,1)$.

It follows from (i) - (iii) that there exist bounded sets V_1, $V_2 \subset R^n$, such that for all trajectories $x(\cdot)$ of (1) $x(t) \in V_1$ and for all $y \in F(x(t), t)$, $t \in [0,1]$ $y \in V_2$. Hence we may assume that F is Lipshitzean in x with a constant $L=L(V_1 \cup V_2)$.

For the map $F: V_1 x [0,1] \to 2^{V_2}$ we introduce the averaged modulus of continuity in the following way: Denote by $\omega(F,x,t_0,h)$ the local modulus of continuity of $F(x,\cdot)$. Then

$$\tau(F,h)_p = \| \sup_{x \in V_1} \omega(F,x,\cdot,h) \|_{L_p[0,1]} .$$

The properties of this modulus are the same as 1^o- 5^o, but in 4^o F must have uniformly (in x) bounded p-variation (property (A)) and in 5^o F must be Hausdorff continuous for all $x \in V_1$ and almost all $t \in (0,1)$ (property (B)).

3. <u>Proof of the theorem</u>. Consider the Euler discrete approximation of the inclusion (1), given by (2).

It follows from (i)-(iii) that there exist bounded sets V_3, V_4 such that for any solution of (2) $\{x_i, i=0,..,N\}$ and for $i=1,..,N$ $x_i \in V_3$ and for $y \in F(x_i,t_i)$ $y \in V_4$. Without loss of generality we may assume that $V_1 \equiv V_3$ and $V_2 \equiv V_4$.

Let B is the unit ball in R^n, $g: [0,1] \to R_+$, $\|.\|_C$ denotes the norm in $C(R^n,[0,1])$ and $G(t) = \int_0^t g(s)ds$.

First we give an estimate of the distance between the set of trajectories of (1) and the set of the trajectories of the extended inclusion

(5) $\dot{x}(t) \in F(x(t),t) + g(t)B$ for a.e. $t \in (0,1)$, $x(0) = x^0$.

Lemma 1. If (i)-(iii) hold and $x(\cdot)$ is a solution of (5), then there exists a solution of (1) $x^*(\cdot)$, such that

$\| x - x^* \|_C \le G(1)\exp(L)$.

The Theorem is a direct consequence of Lemmas 2 and 3:

Lemma 2. If (i)-(iii) hold, then there exists a constant c_2, such that for any discrete trajectory $x = \{x_i, i=0,..,N\}$ of (2) there exists a solution $x(\cdot)$ of (1), for that

(6) $\max_{0 \le i \le N} | x(t_i) - x_i | \le c_2(\tau(F,h)+h)$.

Lemma 3. If (i)-(iii) hold and F has convex images, then there exists a constant c_3, such that for any trajectory x of (1) there exists a trajectory x^N of (2), such that

(7) $\max_{0 \le i \le N} | x(t_i) - x_i | \le c_3(\tau(F,h)+h)$.

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