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ON A GAUSSIAN QUADRATURE FORMULA FOR ENTIRE FUNCTIONS OF EXPONENTIAL TYPE

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Let B_{τ} be the linear space of all entire functions of exponential type τ.

Theorem A (see [2]). For $f(z) \in B_{\tau}$, $\tau < 2\sigma$ the following quadrature formula holds

$$(1) \qquad \int_{-\infty}^{\infty} f(x)\,dx = \frac{\pi}{\sigma} \sum_{\nu=-\infty}^{\infty} f\left(\frac{\nu\pi}{\sigma}\right)$$

if $\int_{-\infty}^{\infty} f(x)\,dx$ and $\sum_{\nu=-\infty}^{\infty} f\left(\frac{\nu\pi}{\sigma}\right)$ exist in the sense of Cauchy.

If $f(z) \in B_{2\sigma}$ and $f(x) = O(|x|^{-\delta})$, $\delta > 1$ then (see [2]) the formula (1) holds.

In this paper we are interested in what about the crucial case $f(z) \in B_{2\sigma}$. For this case one can find the following (see [1])

Theorem B. For $f(z) \in B_{2\sigma} \cap L_1$ the formula (1) holds.

A new idea of proving Th.B is by using Sobolev's function $\varphi_\varepsilon(x)$ and $\psi_\varepsilon(x) = \sqrt{2\pi}\,\widehat{\varphi_\varepsilon}(x) = \int_{-\varepsilon}^{\varepsilon} \varphi_\varepsilon(t) e^{-ixt}\,dt$ (for notations see [4]) -

$$f(z)\psi_\varepsilon(z) \in B_{2(\sigma+\frac{\varepsilon}{2})} \cap L_1 \qquad \text{and} \quad f(x)\psi_\varepsilon(x) = O\left(|x|^{-2}\right), \quad |x| \to \infty.$$

Applying (1) for $f(z)\psi_\varepsilon(z)$ and taking the limit when $\varepsilon \to 0_+$ we get

Th.B after using the following inequality (see (4))

$$(2) \quad \sum_{\nu=N}^{\infty}\left|f\left(\frac{\nu\pi}{\sigma+\varepsilon/2}\right)\right| \leqq \int_{\frac{N\pi}{\sigma+\varepsilon/2}}^{\infty}|f'(t)|\,dt + \frac{\sigma+\varepsilon/2}{\pi}\int_{\frac{N\pi}{\sigma+\varepsilon/2}}|f(t)|\,dt .$$

Proposition 1. Let $I^{\varepsilon}_{\tau,2\sigma}=[\tau,\tau+\varepsilon)\cap[\tau,2\sigma]$ and $J^{\varepsilon}_{\tau,2\sigma}=\{[\frac{\tau}{2},\frac{\tau+\varepsilon}{2})\cap[\sigma,\sigma+\frac{\varepsilon}{2})\}\cup\sigma$ and $f(z)\in B_{\gamma}$ for $\gamma\in I^{\varepsilon}_{\tau,2\sigma}$ and $\int_{-\infty}^{\infty}f(x)dx$ exists in the sense of Cauchy. Then if we consider the following linear spaces -

$$(3) \quad \Omega^{\varepsilon}_{\tau,2\sigma}=\left\{f(x): Q(f,\alpha)\in C\left(J^{\varepsilon}_{\tau,2\sigma}\cap(\alpha:\alpha\geqslant\sigma/2)\right) \text{ with respect to } \alpha\right\}, \text{where}$$

$$Q(f,\alpha)=\sum_{\nu=-\infty}^{\infty}f\left(\frac{\nu\pi}{\alpha}\right) \quad \text{and}$$

$$(4) \quad K^{\varepsilon}_{\tau,2\sigma}=\left\{f(x): \int_{-\infty}^{\infty}f(x)dx=\frac{\pi}{\alpha}Q(f,\alpha) \text{ holds for } \alpha\in J^{\varepsilon}_{\tau,2\sigma}\cap(\alpha:\alpha\geqslant\sigma/2)\right\}$$

then under the above conditions $\Omega^{\varepsilon}_{\tau,2\sigma}$ and $K^{\varepsilon}_{\tau,2\sigma}$ are identical. Using this trivial proposition a nontrivial example will be given. We are interested of $\tau=2\sigma$. Then Prop.1 says that if -

1) $f(z)\in B_{2\sigma}$, 2) $\int_{-\infty}^{\infty}f(x)dx$ exists in the sense of Cauchy, 3) $Q(f,\alpha)\in C[\sigma,\sigma+\frac{\varepsilon}{2})$ then the formula (1) holds for $f(z)$.

Theorem 1. For $\varphi(z)=\sum_{k=1}^{\infty}C_k\frac{\sin\tau_k z}{z}$ where $C_k>0$, $\sum_{k=1}^{\infty}C_k<\infty$ and $\tau_k<\tau_{k+1}$, $\tau_k\in[2\sigma-\delta,2\sigma)$ and $\tau_k\to2\sigma$, $k\to\infty$ we have -

a) $\varphi(z)\in B_{2\sigma}\cap L_2$;

b) $Q(f,\alpha)\in C[\sigma,\sigma+\varepsilon)$;

c) $\varphi(z)\notin B_{\tau}$ for $\tau<2\sigma$;

d) the representation $\varphi(z)=\varphi_1(z)+\varphi_2(z)$ where $\varphi_1(z)\in B_{\tau}$, $\tau<2\sigma$ and $\varphi_2(z)\in B_{2\sigma}\cap L_1$ is impossible.

Corollary 1. From Th.1 and Prop.1 we get that (1) holds for the function $\varphi(z)$.

Proof of Th.1. a) it follows by Weierstrass' theorem, Bernstein's inequality for $B_{2\sigma} \cap L_\infty$ and the L_2-norm inequality.

b) let us consider

$$(5)\quad Q(\varphi,\gamma)=\sum_{\ell=-\infty}^{\infty}\sum_{k=1}^{\infty} c_k \frac{\sin \tau_k \frac{\ell\pi}{\gamma}}{\frac{\ell\pi}{\gamma}} = \frac{\gamma}{\pi}\lim_{N\to\infty}\sum_{\ell=-N}^{N}\sum_{k=1}^{\infty} c_k \frac{\sin \tau_k \frac{\ell\pi}{\gamma}}{\ell}.$$

Because of absolute summability of the series we have

$$(6)\quad \sum_{\ell=-N}^{N}\sum_{k=1}^{\infty} c_k \frac{\sin \tau_k \frac{\ell\pi}{\gamma}}{\ell} = \sum_{k=1}^{\infty} c_k \sum_{\ell=-N}^{N} \frac{\sin \tau_k \frac{\ell\pi}{\gamma}}{\ell}.$$

From (6) we get -

$$(7)\quad Q(\varphi,\gamma)=\lim_{N\to\infty}\frac{\gamma}{\pi}\sum_{\ell=-N}^{N}\sum_{k=1}^{\infty} c_k \frac{\sin \tau_k \frac{\ell\pi}{\gamma}}{\ell} = \lim_{N\to\infty}\frac{\gamma}{\pi}\sum_{k=1}^{\infty} c_k \sum_{\ell=-N}^{N} \frac{\sin \tau_k \frac{\ell\pi}{\gamma}}{\ell}.$$

Because of

$$(8)\quad \left|\sum_{\ell=1}^{N}\frac{\sin \ell x}{\ell}\right| \le const \qquad \text{for every } x \text{ (here } 0<\frac{\tau_k\pi}{\gamma}<2\pi$$

we get the following -

$$(9)\quad \lim_{N\to\infty}\sum_{k=1}^{\infty} c_k \sum_{k=-N}^{N}\frac{\sin \tau_k \frac{\ell\pi}{\sigma}}{\ell} = \sum_{k=1}^{\infty} c_k \sum_{\ell=-\infty}^{\infty}\frac{\sin \tau_k \frac{\ell\pi}{\sigma}}{\ell} = \pi\sum_{k=1}^{\infty} c_k$$

by using Th. A for every function $\frac{\sin \tau_k x}{x}$, $k=1,2,3,\dots$.

c) Because of

$$\left|\sum_{k=1}^{\infty} c_k \frac{\sin \tau_k v}{v} e^{-ivx}\right| \le \sum_{k=1}^{\infty} c_k \tau_k \in L_1[-N,N]$$

we have by the Lebegue's theorem -

$$(10)\quad \int_{-N}^{N}\varphi(v)e^{-ivx}dv = \sum_{k=1}^{\infty} c_k \int_{-N}^{N}\frac{\sin \tau_k v}{v}e^{-ivx}dv.$$

From (10) we get the following -

$$(11)\quad \int_{-\infty}^{\infty}\varphi(v)e^{-ivx}dv=\lim_{N\to\infty}\int_{-N}^{N}\varphi(v)e^{-ivx}dv=\lim_{N\to\infty}\sum_{k=1}^{\infty}C_k\int_{-N}^{N}\frac{\sin\tau_k v}{v}e^{-ivx}dv.$$

But we have the following -

$$(12)\quad \left|\int_{-N}^{N}\frac{\sin\tau_k v}{v}e^{-ivx}dv\right|=\left|\int_{-N}^{N}\frac{\sin\tau_k v}{v}\cos vx\,dv\right|\leqq$$

$$\frac{1}{2}\left|\int_{-N|\tau_k+x|}^{N|\tau_k+x|}\frac{\sin w}{w}dw\right|+\frac{1}{2}\left|\int_{-N|\tau_k-x|}^{N|\tau_k-x|}\frac{\sin w}{w}dw\right|.$$

By (12) for $\tau_k\in[2\sigma-\delta,2\sigma)$ and x - fixed number (real) we get -

$$(13)\quad \int_{-N}^{N}\frac{\sin\tau_k v}{v}e^{-ivx}dv\leqq const(x)\ \text{for}\ N\geqslant N_0\ .$$

Using (13) and by (11) we get -

$$(14)\quad \sqrt{2\pi}\ \widehat{\varphi}(x)=\sum_{k=1}^{\infty}C_k\int_{-\infty}^{\infty}\frac{\sin\tau_k v}{v}e^{-ivx}dv$$

and applying that

$$(15)\quad \sqrt{2\pi}\ \widehat{\frac{\sin\tau_k(\cdot)}{(\cdot)}}(x)=\begin{cases}\pi & \text{for } x\in(-\tau_k,\tau_k)\\ \frac{\pi}{2} & \text{for } x=\tau_k,-\tau_k\\ 0 & \text{for } |x|>\tau_k\end{cases}$$

we get the following -

$$(16)\quad supp\ \widehat{\varphi}=[-2\sigma,2\sigma]$$

and this ends the proof of c).

d) let us assume that

(17) $\varphi(z)=\varphi_1(z)+\varphi_2(z)$, where $\varphi_1(z)\in B_\tau$, $\tau<2\sigma$ and $\varphi_2(z)\in B_{2\sigma}\cap L_1$.

Then because of $\varphi_2(z)\in L_2$ and $\varphi(z)\in L_2$ we obtain $\varphi_1(z)\in L_2$.

and from here

(18) $supp\ \widehat{\varphi}_1\in[-\tau,\tau]$ from Paley - Wiener's theorem. But

$\widehat{\varphi}_2(x) \in C(R)$ and we get

$$(19) \qquad \widehat{\varphi}(x) \in C\left(R\setminus[-\tau,\tau]\right)$$

but this is impossible because of (14) and (15).

Remarks:

1. Let us consider $\overline{\left(\bigcup_{\tau<2\sigma} B\tau\right)}_C = A_{2\sigma}$ - the closure in C - metric.

It is evident that $\varphi(z)\in A_{2\sigma}$. We ask about the following question- if for every $f(z)\in A_{2\sigma}$ the quadrature formula (1) holds?

For $\varphi(z)\in A_{2\sigma}$ the formula (1) holds. But for the function $\frac{\sin 2\sigma z}{z}$ and the functions $\frac{\sin \tau_k z}{z}$, $\tau_k<\tau_{k+1}$, $\tau_k \to 2\sigma$ we have -

a) $\frac{\sin \tau_k x}{x} \xrightarrow[C]{} \frac{\sin 2\sigma x}{x}$; b) $\frac{\sin\tau_k x}{x} \in \bigcup_{\tau<2\sigma} B\tau$; c) $\frac{\sin 2\sigma x}{x} \in A_{2\sigma}$ but

$$(20) \quad \pi = \int_{-\infty}^{\infty} \frac{\sin 2\sigma x}{x}\,dx \neq \frac{\pi}{\sigma}\sum_{\ell=-\infty}^{\infty} \frac{\sin 2\sigma \frac{\ell\pi}{\sigma}}{\ell\frac{\pi}{\sigma}} = 2\pi .$$

2. Another idea of proving Th.B can be based on Prop.1 and the inequality (2).

3. For $f(x)=\frac{\sin 2\sigma x}{x}$ the function $Q(f,\alpha)$ is continuous for $\alpha\in[\sigma+\varepsilon_1,\sigma+\varepsilon)$ for $\varepsilon_1>0, \varepsilon_1<\varepsilon$ but not for $\alpha\in[\sigma,\sigma+\varepsilon)$-

$$(21) \quad Q(f,\alpha) = \sum_{\ell=-\infty}^{\infty} \frac{\sin 2\sigma\frac{\ell\pi}{\alpha}}{\frac{\ell\pi}{\alpha}} = \frac{\alpha}{\pi}\left(\frac{2\sigma\pi}{\alpha} + 2\sum_{\ell=1}^{\infty}\frac{\sin\frac{2\sigma}{2\alpha}2\pi\ell}{\ell}\right),$$

and

$$(22) \qquad \sum_{\ell=1}^{\infty}\frac{\sin\ell x}{\ell} = \begin{cases} 0 & \text{for } x=2K\pi \\ \frac{\pi-x}{2} & \text{for } x\neq 2K\pi .\end{cases}$$

4. The analogous consideration can be made for the qudrature formula of Turan's type (see [3]) -

$$(23) \quad \int_{-\infty}^{\infty} f(x)\,dx = \frac{\pi}{\sigma}\sum_{\ell=0}^{\frac{m-1}{2}} \frac{1}{(2\sigma)^{2\ell}}\, C_{\ell,m-1}\sum_{\nu=-\infty}^{\infty} f^{(2\ell)}\left(\frac{\nu\pi}{\sigma}\right)$$

m - odd, which is roughly speaking exact for $B_{(m+1)\sigma}$.

References

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