CONSTRUCTIVE THEORY OF FUNCTIONS'87

SOFIA, 1988

APPROXIMATION OF PERIODIC FUNCTIONS OF ONE AND SEVERAL VARIABLES

E.M. Galeev

Introduction. The paper deals with some problems of approximation for classes of periodic functions of one and several variables and finite dimensional sets. The subjects of approximation are the classes with bounded mixed derivative $\tilde{W}^r_p$ and their intersections $\tilde{W}^{\alpha} = \tilde{W}^{\bar{r}}_{\bar{p}} = \bigcap_{i=1}^{m} \tilde{W}^{r^i}_{p^i}$, the classes $\tilde{H}^r_p$, O.V. Besov classes $B^r_{p,\theta}$ and finite dimensional balls B^m_p. The paper contains new results concerning Kolmogorov and Bernstein diameters of these classes. We also introduce and calculate the spectral diameters of finite dimensional sets, the nuclear diameters of functional classes.

Orders of norms for the derivatives of Dirichlet and Favard kernels with N harmonics having the least value are defined. For the previous results and more detailed references see [1] - [5].

We introduce the notation. Let $T^n = [-\pi, \pi]^n$ be n-dimensional torus, and let $\tilde{L}_p = \tilde{L}_p(T^n)$, $1 \le p \le \infty$, denote the space of functions $x(t) = x(t_1, \dots, t_n)$, 2π-periodic with respect to each variable, with the norm $\|x\|_p = \|x(\cdot)\|_{L_p}$.

To simplify these statements, let's restrict ourselves to the functions whose Fourier coefficients having at least one zero index are equal to zero, i.e. $x(t) = \sum_{k \in \mathring{Z}^n} x_k e^{i(k,t)}$, where $\mathring{Z}^n = \{k = (k_1, \dots, k_n)$, k_i is integer, $k_i \ne 0$, $i = 1, \dots, n\}$. For these functions and vector $r \in R^n$ we introduce the operation of fractional differentiation by the formula $x^{(r)}(t) = \sum_k (ik)^r \cdot x_k e^{i(k,t)}$.

We relate every vector $s \in N^n$ to the subset $\square_s \subset \mathring{Z}^n$ according to the following rule:

Then $x(t) = \sum_k x_k e^{i(k,t)} = \sum_s \delta_s x(t)$, where $\delta_s x(t) = \sum_{k \in \square_s} x_k e^{i(k,t)}$.

For vector $r \in R^n$ and numbers $1 \le p \le \infty$, $1 \le \theta \le \infty$, we define the functional classes $\tilde{W}^r_p = \{x(\cdot) \mid \|x^{(r)}\|_p \le 1\}$, $B^r_{p,\theta} = \{x(\cdot) \mid \|x\|_{B^r_{p,\theta}} = \left(\sum_s \|\delta_s x^{(r)}\|_p^\theta\right)^{1/\theta} \le 1\}$ and $\tilde{H}^r_p = B^r_{p,\infty}$

1. Spectral diameters of the finite dimensional sets. Estimates of the diameters of functional classes often reduce to estimates of the diameters of finite dimensional sets. For example, the lower estimate of orthoprojective diameters of functional classes can reduce to the estimate of orthoprojective diameters of finite dimensional sets. It is convenient to give lower estimates of these diameters as well as projective diameters $\pi_N(W, X)$ by spectral diameters introduced below.

Definition. As the spectral N-diameter of a set W in Banach space X we shall call the quantity

$$tr_N(W, X) = \inf_T \sup_{x \in W} \|x - Tx\|_{X'}$$

where infimum is taken over the finite dimensional linear continuous operators $T : X \to X$, for which the trace of operator equals N ($tr\, T = N$).

Note that the definition of diameters is not directly connected with dimension; and N can also take fractional values. Consider, for example, some theorems of the spectral and projective diameters of unit ball B^m_p in space l^m_q and the set V_k, $1 \le k \le m$, which is a convex hull in R^m of points with any k coordinates equal to ± 1, and the others being zeroes.

Theorem 1.1. Let $1 \le q \le \infty$, $N \in R$, $m \in N$, $1 \le k \le m$. Then

$$tr_N(V_k, l^m_q) = k^{\frac{1}{q}} \left|1 - \frac{N}{m}\right|.$$

Theorem 1.2. Let $1 \le p, q \le \infty$, $N \in R$, $m \in N$. Then

$$tr_N(B^m_p, l^m_q) = \left|1 - \frac{N}{m}\right| m^{\left(\frac{1}{q} - \frac{1}{p}\right)_+}$$

Theorem 1.3. The following statement holds:

$$\pi_N(B^m_1, l^m_\infty) = 1 - \frac{N}{m}, \quad N \le m.$$

2. Nuclear diameters of the functional classes. V.N.Temlyakov proved that the Fourier operator gives for $1 < p \le q < \infty$ the best approximation among the orthoprojective operators and among the linear operators with restriction

$$\| P e^{i(k,\cdot)} \|_2 \le C \quad \forall k \in \overset{\circ}{Z}{}^n \quad (C \ge 1).$$

Let us give other conditions on linear operators under which the Fourier operator gives the optimal (in the sense of order) approximation without any additional space $\tilde{L}_2$ and the concept of orthogonality that makes sense only in a Hilberbert space.

Definition. As the nuclear N -diameter of a set W in a Banach space X we shall call the quantity

$$\mathcal{N}_N(W,X) = \inf_{P \in \mathcal{P}(N,X)} \sup_{x \in W} \| x - Px \|_X ,$$

where $\mathcal{P}(N,X)$ is the set of nucler operators P with finite nuclear norm $N(P) \le N$ $(N > 0)$.

Note that the definition of the nuclear diameter as well as the definition of the spectral diameter is not directly connected with dimension; and N can also take fractional values.

It is obvious that if P is the orthoprojector onto the subspace of dimension N in the space $\tilde{L}_2$ then $P \in \mathcal{P}(N, \tilde{L}_2)$. It is not difficult to prove that if S_N is the Fourier operator in the space $\tilde{L}_q$, acting onto the subspace of dimension N, then $S_N \in \mathcal{P}(N^q, \tilde{L}_q)$.

In the approximation theory of classes $\tilde{W}^{\mathfrak{A}} = \bigcap_{i=1}^{m} \tilde{W}^{r^i}_{p^i}$, $\mathfrak{A} = \{(\frac{1}{p^i}, r^i), i = 1,\dots,m\}$, $1 < p^i < \infty$, $r^i \in R^n$, sets G play an important part where $G = G(\mathfrak{A}) = \mathrm{conv}\, \mathfrak{A} + \mathrm{cone}\{(e_j, 0), (-e_j, -e_j), j = 1,\dots,n\}$, and $e_1,\dots,e_n$ is the canonical basis in R^{n}, $G_q = \{ r \in R^n \mid (\frac{1}{q}, r) \in G \}$; so does the function $\gamma(\frac{1}{q}) = \max\{ r \mid (\frac{1}{q}, r) \in G(\mathfrak{A}) \}$ for classes $\tilde{W}^{\mathfrak{A}}(T^1)$. For a set $A \subset R^n$ let us introduce the following notation $S(A) = \{ s \in R^n_+ \mid (s, \alpha) \le 1 \ \forall \alpha \in A \}$. For a vector $\beta \in R^n$ and a set $S \in R^n$ we denote by $M(\beta, S)$ and $\nu(\beta, S)$ the value and the dimension of the affine hull of the set of solutions of the following problem: $(s, \beta) \to \sup;\ s \in S$ (the values M and ν arise in calculating the number of integer points in a logarithmically polyhedral set (see [4])).

Theorem 2.1. Let $\tilde{W}^{\mathfrak{A}} = \tilde{W}^{\mathfrak{A}}(T^1)$, $1 < p^i, q < \infty$, $r^i \in R$, $i = 1,\dots,m$, $\gamma(\frac{1}{q}) > 0$. Then

$$\mathcal{N}_N(\tilde{W}^{\alpha},\tilde{L}_q)\asymp N^{-\gamma(\frac{1}{q})}$$

Theorem 2.2. Let $\tilde{W}^{\alpha}=\tilde{W}^{\alpha}(T^n)$, $1<p^i,q<\infty$, $r^i\in R^n$, $i=1,\dots,m$, $G_q\cap \overset{\circ}{R}{}^n_+\neq\emptyset$, $M,\nu=M,\nu(\mathbb{1},S(G_q))$. Then

$$\mathcal{N}_N(\tilde{W}^{\alpha},\tilde{L}_q)\asymp(N^{-1}\log^{\nu}N)^{\frac{1}{M}}$$

Theorem 2.3. Let $1<q\le p\le 2$, $1\le\theta\le\infty$, $r\in R^n$, $0<r_1=\dots=r_{\nu+1}<r_{\nu+2}\le\dots\le r_n$. Then

$$\mathcal{N}_N(B^r_{p,\theta},\tilde{L}_q)\asymp(N^{-1}\log^{\nu}N)^{r_1}(\log^{\nu}N)^{(\frac{1}{p}-\frac{1}{\theta})_+}$$

Upper estimates in all the theorems are obtained by approximating the Fourier operator consisting of N harmonics. The description of the Fourier operator is in the papers [2], [3].

3. Kolmogorov diameters.

Theorem 3.1. Let $1<p^i<\infty$, $r^i\in R$, $i=1,\dots,m$, $\tilde{W}^{\alpha}=W^{\alpha}(T^1)$. Then

$$d_N(\tilde{W}^{\alpha}_p,\tilde{L}_q)\asymp\begin{cases}N^{-\gamma(\frac{1}{q})}, & 1<q\le 2,\ \gamma(\frac{1}{q})>0,\\ N^{-\gamma(\frac{1}{2})}, & 2\le q<\infty,\ \gamma(\frac{1}{2})>\frac{1}{2}.\end{cases}$$

Theorem 3.2. Let $1<p\le q\le 2$, $r\in R^n$, $\frac{1}{p}-\frac{1}{q}<r_1=\dots=r_{\nu+1}<r_{\nu+2}\le\dots\le r_n$. Then

$$d_N(\tilde{H}^r_p,\tilde{L}_q)\asymp(N^{-1}\log^{\nu}N)^{r_1-\frac{1}{p}+\frac{1}{q}}\log^{\frac{\nu}{q}}N.$$

Lower estimates in both theorems reduce to lower estimates of Kolmogorov diameters of finite dimensional sets through discretization. The lower estimates of the finite dimensional diameter is based on the averaging of the distance up to the subspace of dimension N over the vertices of the set obtained.

4. Bernstein diameters. Remember that the Bernstein N-dia-

meter of the centrally symmetric set W in a normed space X with a unit ball B is the quantity

$$b_N(W,X) = \sup_{\varepsilon, L_N} \{\varepsilon \mid \varepsilon B \cap L_N \subset W\},$$

where L_N is the subspace of dimension N in X.

The calculation of Bernstein diameters can reduce to the calculation of Kolmogorov codiameters according to the following formula: $b_N(W,X) = (d_{-N}(W^{\circ}, X^{*}))^{-1}$ (to define the codiameter, the exterior infimum in the definition of d_N should be taken over all possible subspaces of the codimension N), where W° is a polar of the set W.

Theorem 4.1. Let $1 < p,q < \infty$, $r \in R^n$, $0 < r_1 = \dots = r_{\nu+1} < r_{\nu+2} \le \dots \le r_n$. Then

$$b_N(\tilde W^r_p, \tilde L_q) \asymp \begin{cases} (N^{-1}\log^{\nu} N)^{r_1+\frac{1}{q}-\frac{1}{p}}, & 2 \le q \le p,\ r_1 - \frac{1}{p} + \frac{1}{q} > \frac{1}{2}; \\ (N^{-1}\log^{\nu} N)^{r_1+\frac{1}{2}-\frac{1}{p}}, & q \le 2 \le p,\ r_1 > \frac{1}{p}; \\ (N^{-1}\log^{\nu} N)^{r_1}, & p \le q;\ q \le p \le 2,\ r_1 > \frac{1}{\min(p,2)} - \frac{1}{\max(q,2)}; \end{cases}$$

$b_N(\tilde H^r_p, \tilde L_q) \asymp b_N(\tilde W^r_p, \tilde L_q)$ if $p,q \le 2, r_1 > \frac{1}{2}$; $2 \le q \le p, r_1 - \frac{1}{p} + \frac{1}{q} > \frac{1}{2}$; $r_1 > \frac{1}{p}$, $q \le 2 \le p$.

Note the exact values of the diameters of the infinite dimensional parallelipiped $B_\infty(\frac{1}{r}) = \{x = (x_1, \dots, x_n, \dots) \mid |x_k| \le r_k \ \forall k \in N\}$, $r \in l_2$, $r_1 \ge r_2 \ge \dots \ge r_n \ge \dots > 0$

$$b_N(B_\infty(\tfrac{1}{r}), l_2) = \min_{0 \le m < N} \Big(\sum_{k=m+1}^{\infty} r_k^2 / (N-m) \Big)^{1/2},$$

and the infinite dimensional ellipsoid $B_p(r) = \{x = (x_1, \dots, x_n, \dots) \mid \sum_{k=1}^{\infty} |x_k r_k|^p \le 1\}$, $r = (r_1, \dots, r_n, \dots)$, $0 < r_1 \le r_2 \le \dots \le r_n \le \dots$

$$b_N(B_p(r), l_q) = \left(\sum_{k=1}^{N} r_k^{\frac{pq}{q-p}}\right)^{\frac{p-q}{pq}}, \quad 1 \le p \le q \le \infty$$

The theorem 4.1 is the continuation of Maiorov's research-work who defined the orders of the Bernstein diameters $b_N(\tilde{W}_p^r, \tilde{L}_q)$ of the classes of periodic functions of one variable for $q \le p$. The author was informed of Tzarkov having defined the orders $b_N(\tilde{W}_p^r, \tilde{L}_q)$ of the classes of periodic functions of one variable for $p \le q$. In some cases he obtained the exact values for the diameters.

5. The orders of norms for the derivatives of Dirichlet and Favard kernels with N harmonics having the least value are defined. In the approximation theory of importance are Dirichlet and Favard kernels. Let us define the quantities

$$L_N(r,q) = \inf_{K_N} \| (\sum_{k \in K_N} e^{i(k,\cdot)})^{(r)} \|_q,$$

$$F_N(r,q) = \inf_{K_N} \| (\sum_{k \in \mathring{Z}^n \setminus K_N} e^{i(k,\cdot)})^{(-r)} \|_q,$$

(where $K_N \subset \mathring{Z}^n$ is an arbitrary set of N harmonics), which for functions of one variable were introduced and calculated by V.E. Maiorov. Applying Maiorov's proof to the functions of one variable we defined the orders of the corresponding quantities for the functions of several variables.

Theorem 5.1. Let $r \in R^n$, $r_1 = \ldots = r_{\nu+1} < r_{\nu+2} \le \ldots \le r_n$, $1 < q < \infty$. Then $L_N(r,q) = 0$ if $r_1 < 0$ and if $r_1 \ge 0$, then

$$L_N(r,q) \asymp N^{r_1+1-\frac{1}{q}} / (\log^{\nu} N)^{(r_1+1-\frac{2}{q})_+}, \quad q > 2,\ r_1 \ge \frac{1}{q} \text{ or } q \le 2,\ r_1 \ge 0;$$

$$L_N(r,q) \asymp N^{\frac{1}{2}}, \quad q > 2,\ r_1 = 0;$$

$$L_N(r,q) \gg N^{\frac{r_1 q+1}{2}} / (\log^{\nu} N)^{\frac{r_1 q+1}{2} - \frac{1}{q}}, \quad q > 2, \ 0 < r_1 \le \frac{1}{q}.$$

Theorem 5.2. Let $r \in R^n$, $r_1 = \ldots = r_{\nu+1} < r_{\nu+2} \le \ldots \le r_n$, $1 < q < \infty$. Then the quantity $F_N(r,q)$ is finite if and only if $r_1 > 1 - \frac{1}{q}$; and if $r_1 > 1 - \frac{1}{q}$, then

$$F_N(r,q) \asymp (N \log^{-\nu} N)^{-r_1+1-\frac{1}{q}} \log^{\frac{\nu}{q}} N$$

References

1. V.M.Tihomirov. Some questions of approximation theory. Izdat. Moskov. Gos. Univ., Moscow, 1976.
2. E.M.Galeev. The approximation of classes of functions with several bounded derivatives by Fourier sums. Mat. Zametki, v. 23, 1978, p. 197-211.
3. V.M.Temlyakov. Approximation of functions with bounded mixed derivative. Trudy Mat. Inst. Steklov, v. 178, 1986, p. 3-112.
4. Din' Zung. The approximation for classes of smooth functions of several variables. Trudy sem. Petrovskogo, № 10, 1984, p. 207-226.
5. V.E.Maiorov. Trigonometric diameters of Sobolev classes. Theory of functions and approximations. Trudy of winter school in Saratov. Izdat. Saratov. Izdat. Saratov Gos. Univ., Saratov, 1986, part 3, p. 20-22.

Department of Mechanics and Mathematics
Moscow State University
119899 Moscow USSR