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FROISSART DOUBLETS IN THE PADE APPROXIMATION AND NOISE

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Introduction

We present further work on a twenty year old problem around curious and interesting Froissart numerical experiments. Our contribution concerns two different aspects. On the one hand we prove by standard probability methods that the unit circle is a natural boundary, with probability 1, of the noise function, which will be defined later. On the other hand we take advantage of Froissart doublets to enable us to complete by a simple method the well known algorithms for the choice of the "best Padé approximant" [7,8,9]. Remember that the $[m/n]_f(z)$ Padé approximant to the series (function) f is a rational function $P_m(z)/Q_n(z)$ defined such that its Maclaurin series agrees with the initial series f up to the order m+n, and where P_m and Q_n are the polynomials of degrees m and n respectively.

Froissart results

Two important questions arise in the numerical applications (where we only have a finite amount of information) of the Padé approximation method:

(i) what is the best Padé approximant in the finite and calculable set of approximants ?

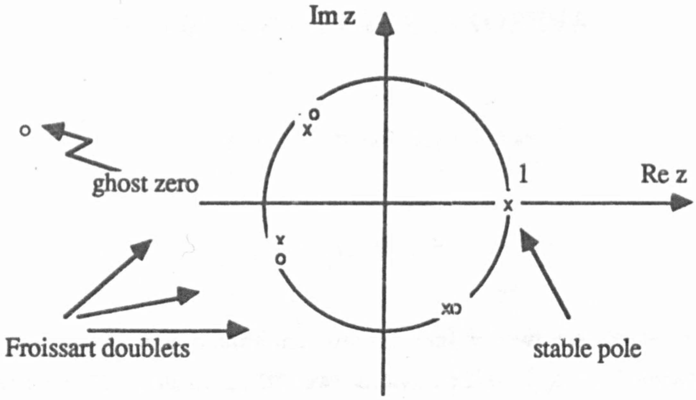
(ii) what can we say about the stability of the above approximant ?

Several problems of theoretical physics lead to the summation of Stieltjes series. In this case the best Padé approximants [7] belong to the staircase: [n-1/n], [n/n]. The main problem in this case is the numerical stability of the Padé approximants with respect to numerical or experimental noise on the series coefficients. For instance, there are always computer errors which are related to the floating point representation; in the problem of interpolation by rational functions of some experimental data (fit problem) we have to take into account experimental errors. Twenty years ago Froissart simulated the numerical noise of the coefficients of power series of analytic functions. His results can be compiled as follows.

Consider the power series

$$F(z) = f(z) + g(z) = \sum_{n=0}^{\infty} (1+ \varepsilon r_n)\ z^n$$

where f represents the analytical signal $\frac{1}{1-z}$ and g a non-analytical noise: here, and in the sequel, r_n are random complex numbers such that $|r_n| \leq 1$ and ε is a small parameter (for instance of order 10^{-3}). It will be shown that $|z|=1$ is the natural boundary for the noise g with a probability 1. Computing the poles and zeros of [n/n] Padé approximants to F, Froissart observed the following:



$$[4/4]_F(z) = \frac{(z+\ 55\ \ -\ \ \ 5i\ )\ (z+.95+.21i)\ (z+.575-.805i)\ (z-.453+.754i)}{(z-1.003+.0008i)\ (z+.97+.19i)\ (z+.572-.791i)\ (z-.457+.749i)}$$

(i) there is a very stable pole near z=1 (true pole of f) and the error made on the position is of order ε for [1/1] and decreases as n increases;

(ii) there is an unstable (as the order n changes) zero at a distance of order $1/\varepsilon$;

(iii) all the other poles and zeros group themselves in the vicinity of the unit circle in pairs ("doublets"): one pole and one zero separated by a distance of order ε, unstable with respect to the order n and representing the natural boundary of the noise;

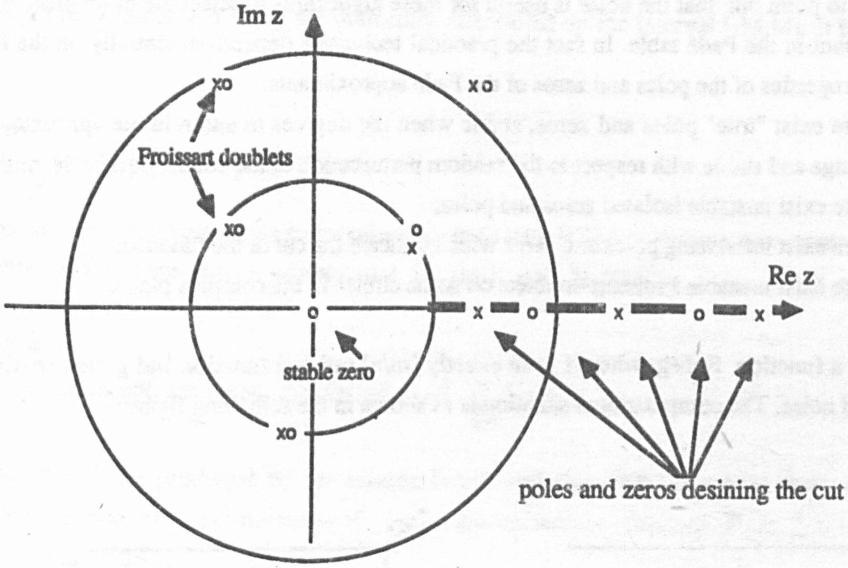
(iv) staying well away from the doublets, the Padé approximants of F are very close to f within ε, even if z is large.

The second example is given by the following series:

$$F(z) = \sum_{n=1}^{\infty} \left(-\frac{1}{n} + \frac{\varepsilon r_n}{2^n} \right) z^n \ .$$

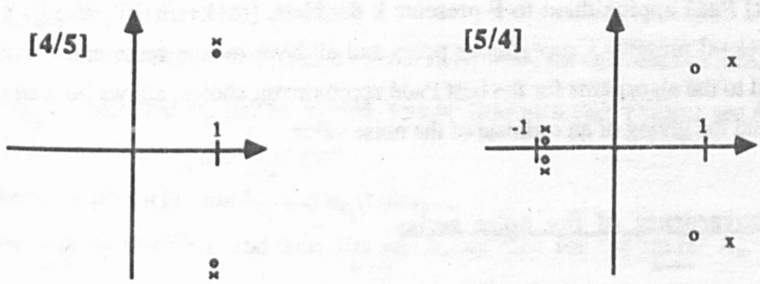
The "regular" part representing ln(1-z) converges for |z|<1 and the noise converges for |z|<2.

However, we observe two noise circles as shown in the following figure:



Unfortunately the Froissart doublets appear also in some "regular" cases as shown by the author [7,p.420] in the example of the series representing the drag coefficient of two spheres in a liquid:

$$f(d) = 1 - \frac{3}{4}d + \frac{9}{16}d^2 - \frac{19}{64}d^3 + \frac{93}{256}d^4 - \frac{327}{1024}d^5 + \frac{1197}{4096}d^6 - \frac{5331}{16384}d^7 + \frac{19821}{65536}d^8 - \frac{76115}{262144}d^9 + \ldots$$



But these doublets appear and disappear when the order of approximation changes and their position has not a significant structure.

In addition we remark that the excessive use of the diagonal approximants (without justification) sometimes leads to accidents, as in the above example [7,pp.414-420] where [2/2](1) proposed by Van Dyke gives 20% of error and the best Padé approximant [1/7](1), deduced from the same information as the starting point of Van Dyke, gives only 0.18% of error.

Froissart doublets as an extra tool for finding the best Padé approximant

Many algorithms are proposed [7,8,9] for the best Padé approximant detection in numerical practice. We wish to point out that the noise is useful for these algorithms to select the exact place of the best approximant in the Padé table. In fact the practical technique depends essentially on the following general properties of the poles and zeros of the Padé approximants:

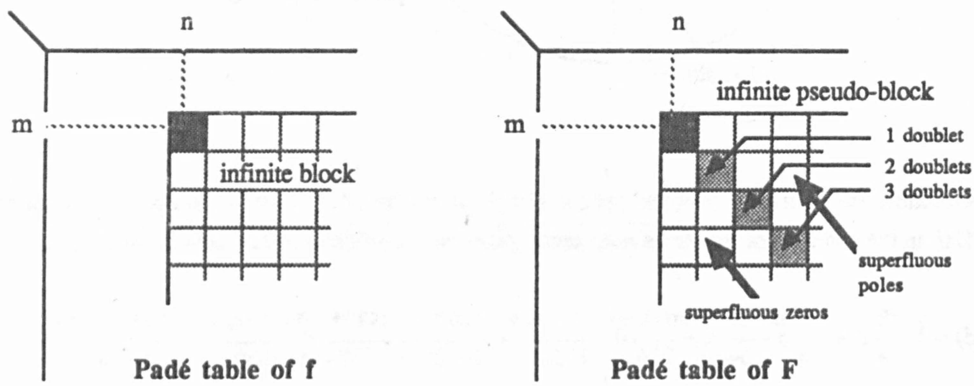
(i) there exist "true" poles and zeros, stable when the degrees m and n in the approximant [m/n] change and stable with respect to the random perturbation of the coefficients of the initial series;

(ii) there exist unstable isolated zeros and poles;

(iii) there exist interlacing poles and zeros which indicate the cut of the function;

(iv) there exist unstable Froissart doublets on some circles in the complex plane.

Consider a function F=f+g where f is an exactly [m/n] rational function and g the experimental or numerical noise. The computational situation is as shown in the following figure:



The [m+k/n+k] Padé approximant to F presents k doublets, [m+k+s/n+k] presents s superfluous zeros, [m+k/n+k+s] presents s superfluous poles and all have m true zeros and n true poles. This analysis, added to the algorithms for the best Padé approximant choice, allows both an improvement of the choice and the giving of an estimate of the noise value.

Radius of convergence of the noise series

Let $\sum_{n=0}^{\infty} r_n z^n$ be a series where r_n are independent random variables in a probability space $(\Omega, \mathcal{A}, P)$.

Note that each ω belonging to Ω may be interpreted as a realisation or a simulation $\{r_n(\omega)\}$ of the sequence $\{r_n\}$. The radius of convergence of the given series depends on ω:

$$\rho(\omega) = [\limsup_{n\to\infty} |r_n(\omega)|^{1/n}]^{-1} .$$

It is easily seen that ρ is then a random variable.

Hypothesis 1

Suppose the probability law P of the random sequence $\{r_n\}$ such that for some positive constant M $|r_n|\leq M$ with probability 1, i.e. $|r_n(\omega)|\leq M$ for ω belonging to some subset Ω_1 in Ω with $P(\Omega_1) = 1$.

For instance this is the case if the r_n are uniformly distributed on the interval [-M,M]. It is obvious that:

Proposition 1

Under hypothesis 1, $\rho\geq 1$ with probability 1.

Now we must show that there exists some subset Ω_2 in Ω with $P(\Omega_2)=1$ and some constant $c>0$ such that for each $\omega\in\Omega_2$ there exists a subsequence $\{r_{k_n}(\omega)\}$ with $|r_{k_n}(\omega)|\geq c$.

Hypothesis 2

Suppose there exists $c>0$ such that $P[|r_n|\geq c]=\delta_n$ and $\sum_{n=0}^{\infty}\delta_n=\infty$.

In practice, the r_n are produced by the unique law P and then $P[|r_n|\geq c]=\delta$. If $\delta>0$ then the hypothesis 2 is satisfied. For instance if $\{r_n\}$ are uniformly distributed on [-1,+1], then $P[|r_n|>c]=1-c$ with $0<c<1$.

Proposition 2

Under hypothesis 2, $\rho\leq 1$ with probability 1.

Proof. Let A_n be the event $[|r_n|\geq c]=\{\omega\in\Omega/|r_n(\omega)|\geq c\}$ for $n\in\mathbf{N}$. Since the random variables $\{r_n\}$ are independent, the $\{A_n\}$ are also independent. We have $\delta_n=P(A_n)$. The following Borel-Cantelli lemma

"*if the events* A_n *are independent and if* $\sum_{n=0}^{\infty}P(A_n)=\infty$ *then* $P[\liminf_{n\to\infty} A_n]=1$ *where* $\liminf A_n=\bigcap_{n=0}^{\infty}\bigcup_{k=n}^{\infty}A_k$."

says that all the A_n simultanuosly occur after a certain rank. Also, for each $\omega\in\liminf_{n\to\infty} A_n$ there exists a subsequence $r_{k_n}(\omega)$ such that $|r_{k_n}(\omega)|\geq c$. Indeed, $\forall n\in\mathbf{N}\ \exists j(n)$ such that $\forall k\geq j(n)$ $\omega\in A_k$; therefore:

for n=0 we take $k_0=j(0)$, $\omega\in A_{k_0}$ and $|r_{k_0}(\omega)|\geq c$,

for n=1 we take $k_1=\sup(k_0,j(1))$, $\omega\in A_{k_1}$ and $|r_{k_1}(\omega)|\geq c$, ...

Finally, we have $\lim_{n\to\infty}|r_{k_n}(\omega)|^{1/n}=1$ and then $\limsup_{n\to\infty}|r_n(\omega)|^{1/n}\geq 1$ for $\omega\in\liminf_{n\to\infty} A_n$. ♦♦♦

Consequently, under both the previous hypotheses $\rho(\omega)=1$ for $\omega\in\Omega_1\cup\Omega_2$ where $\Omega_2=\liminf_{n\to\infty} A_n$.

Then the following result holds:

Proposition

The radius of convergence $\rho=1$ with probability 1.

Conjectures about the Froissart doublets

Because the Froissart noise function presents the natural boundary in the Weierstrass sense, it is plausible to associate this function with some class of quasianalytic functions. Gammel [5,6] conjectured that the Froissart noise function belongs to the Borel class of quasianalytic functions of the form

$$\sum_{n=0}^{\infty} \frac{A_n}{1 - \alpha_n z}$$

where A_n decreases rapidly and the α_n are randomly distributed on the unit circle. If this conjecture is true, then by the Gammel-Nuttall theorem [4] the [n/n] Padé approximants converge in measure in the entire z plane, except on the unit circle, to the noise function. Unfortunately it cannot be proved that the Maclaurin coefficients of the noise function are randomly distributed. Gammel observed himself that for certain sequences $\{A_n\}$ with a prescribed order of diminution, the distribution of those coefficients is peaked at some points and for another type of decreasing sequence of A_n,s it seems to be random. In [7] it was noted that the Borel function chosen by Gammel leads to correlations between the previous coefficients. However other aspects of the theoretical problem of the interpretation of the Froissart doublets and Frossart noise functions remain open.

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