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ON A POLYNOMIAL INEQUALITY CONNECTED WITH THE MULTIVARIATE MARKOV INEQUALITY

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1. Notation. In the following, E stands for a given bounded subset of $\mathbb{R}^N$. We denote by H_n the set of polynomials of N real variables of total degree at most n. Let $A \subset \mathbb{R}^N$ and $Q \in H_n$; we set $\|Q\|_A = \operatorname{Sup}_{x \in A} |Q(x)|$. Classical multivariate notation is used : for $a = (a_1, \ldots, a_N) \in \mathbb{N}^N$, $|a| = a_1 + \ldots + a_N$. For a and k in $\mathbb{N}^N$ $\binom{a}{k} = \prod_{i=1}^{N} \binom{a_i}{k_i}$.

2. Definitions. We say that we have a Markov inequality on E if

(MI) There exist two positive constants C and r such that for any $P \in H_n$

$$\|\partial P/\partial x_i\|_E \leq C\, n^r\, \|P\|_E \quad (i=1,\ldots,N).$$

We know that if E is a Lipschitz set, (MI) holds with $r=2$ [2] and that if

$$E = \{(x,y) \in \mathbb{R}^2 \,/\, 0 \leq x \leq 1 \,,\, 0 \leq y \leq x^p \,,\, p \geq 1\} \,,$$

then (MI) holds with $r=2p$ [1], but the most general known subsets of $\mathbb{R}^N$ for which we have a Markov inequality are uniformly polynomially cuspidal sets recently introduced by Pawlucki and Pleśniak [3]. This is a general class of subsets : Lipschitz sets, bounded convex sets with non-void interior, fat analytic subsets of $\mathbb{R}^N$ are uniformly polynomially cuspidal. (See [3] for details).

For any polynomial R we set

$$d(R) = \operatorname{Sup}_{x \in \bar{E}} \{\operatorname{Inf}\{|a| \,/\, R^{(a)}(x) \neq 0\}\} \,,$$

$$m(R) = \operatorname{Sup}\{m_o > 0 \,/\, \forall x \in \bar{E} \,,\, \exists a \in \mathbb{N}^N \,,\, |a| \leq d(R) \,,\, |R^{(a)}(x)| > m_o\}.$$

We say that we have a polynomial division inequality on E if

(PDI) For any $R \in H_p$, there exist two positive constants C' and r' such that, for any $P \in H_n$ we have $\|P\|_E \le C'(m(R))^{-1}(n+p)^{r'}\|PR\|_E$.

3. Statement of the result. The aim of this paper is to prove the following

Theorem. With previous notation, Markov's inequality implies a polynomial division inequality with $r' = r\,d(R)$.

4. The fundamental lemma. To prove the Theorem we need the following

Lemma. Under assumption (MI), let $d \in \mathbb{N}$. For every $A \subset \overline{E}$ and $R \in H_p$ satisfying

$\{$ There exists $a \in \mathbb{N}^N$ such that $|a| \le d$ and $|R^{(a)}(x)| > m > 0$ $(x \in A)\}$

one can find a positive constant $K(d)$ such that, for any $P \in H_n$,

$$\|P\|_A \le K(d)\, m^{-1}(n+p)^{rd}\,\|PR\|_E + \tfrac{1}{2}\|P\|_E .$$

Proof. We proceed by induction on d.

1) Assume $d=0$. Then $|a| = 0$ and for $x \in A$, $|R(x)| > M$. Therefore, $|P(x)| \le m^{-1}\,|P(x)\,R(x)|$ $(x \in A)$ and

$$\|P\|_A \le m^{-1}\,\|PR\|_A \le K(0)\, m^{-1}\,\|PR\|_E \;(+\tfrac{1}{2}\|P\|_E)$$

with $K(0) = 1$.

2) We now assume that the lemma holds true up to order $d-1$. Let us prove that it is still valid at order d. Let $a \in \mathbb{N}^N$ be such that $|a| = d$ and

$$D = \{k \in \mathbb{N}^N \,/\, |k| > 0 \;,\; 0 \le k_i \le a_i \;(i=1,\ldots,N)\}.$$

From Leibniz' formula, if $x \in A$,

$$|P(x)| \le m^{-1}\{|(PR)^{(a)}(x)| + \sum_{k\in D}\binom{a}{k}|R^{(a-k)}(x)|\,|P^{(k)}(x)|\}.$$

We set $L = \binom{N+d}{d}$ (there are at most $L-1$ elements in D) and

$$A_o = \{x \in A \,/\, |R^{(a-k)}(x)| \le \binom{a}{k}^{-1} m\,\tfrac{1}{2}\,(C\,n^r)^{-|k|}\,L^{-1},\; k \in D\}.$$

If $x \in A_o$ then

$$|P(x)| \le m^{-1}\,|(PR)^{(a)}(x)| + \tfrac{1}{2}L^{-1}\sum_{k\in D}(C\,n^r)^{-|k|}\,|P^{(k)}(x)|.$$

This yields

$$\|P\|_{A_o} \le m^{-1}\,\|(PR)^{(a)}\|_{A_o} + \tfrac{1}{2}L^{-1}\sum_{k\in D}(C\,n^r)^{-|k|}\,\|P^{(k)}\|_{A_o}$$

$$\le m^{-1}\,\|(PR)^{(a)}\|_E + \tfrac{1}{2}L^{-1}\sum_{k\in D}(C\,n^r)^{-|k|}\,\|P^{(k)}\|_E ,$$

and, using (MI)

$$\|P\|_{A_o} \le m^{-1} C^d (n+p)^{rd} \|PR\|_E + \tfrac{1}{2} \|P\|_E$$

(there are less than L terms in Σ).

For every $x \in A \setminus A_o$ there exists $k \in D$ such that

$$|R^{(a-k)}(x)| > \binom{a}{k}^{-1} m \tfrac{1}{2} (C n^r)^{-|k|} L^{-1} . \qquad (1)$$

Then, one can share $A \setminus A_o$ into (at most $L-1$) subsets $A_1, A_2, \ldots$ such that, for every $x \in A_i$ there exists an index k for which (1) holds true. Now, by induction, since $|k| > 0$, on each A_i , we have

$$\|P\|_{A_i} \le (C n^r)^{|k|} L \binom{a}{k} 2 m^{-1} K(d-1) (n+p)^{r(d-|k|)} \|PR\|_E + \tfrac{1}{2} \|P\|_E .$$

Due to the fact that $\|P\|_A = \max_{i=0,1,\ldots} \{\|P\|_{A_i}\}$ we get

$$\|P\|_A \le K(d)\, m^{-1} (n+p)^{rd} \|PR\|_E + \tfrac{1}{2} \|P\|_E$$

with $K(d) = 2 C^d L K(d-1) \max_{a,k,|a|=d} \{\binom{a}{k}\}$ which proves the induction step.

5. Proof of the Theorem. Assume that (MI) holds on E and let $R \in H_p$. For $P \in H_n$ let x be such that $|P(x)| = \|P\|_E$. Then applying the Lemma with $A = \{x\}$, $d = d(R)$, $m = m(R)$ yields

$$\|P\|_E = |P(x)| \le K(d(R))(m(R))^{-1} (n+p)^{r\, d(R)} \|PR\|_E + \tfrac{1}{2} \|P\|_E$$

and therefore

$$\|P\|_E \le 2 K(d(R))(m(R))^{-1} (n+p)^{r\, d(R)} \|PR\|_E .$$

6. Remark. Assume that $E = [-1,1]^N$. Then it is not difficult to show that (PDI) implies (MI).

Proof. Assume that (PDI) holds on E. Let $P \in H_n$ and $a_1 \in [-1,1]$. We define a polynomial Q by

$$Q(x_1,\ldots,x_N) = [P(x_1,\ldots,x_N) - P(a_1,x_2,\ldots,x_N)] / (x_1 - a_1) \quad (x_1 \ne a_1) ,$$

$$Q(a_1,x_2,\ldots,x_N) = \partial P / \partial x_1 (a_1,x_2,\ldots,x_N).$$

Now, applying (PDI) with $R(x) = (x_1 - a_1)$ (and therefore $d(R) = 1$, $m(R) = 1$) gives

$$\|Q\|_E \le C' n^{r'} \|P(x_1,\ldots,x_N) - P(a_1,x_2,\ldots,x_N)\|_E$$
$$\le 2 C' n^{r'} \|P\|_E .$$

In particular,

$$|\partial P/\partial x_1 (a_1,x_2,\ldots,x_N)| \le \|Q\|_E \le 2C' n^{r'} \|P\|_E .$$

The last estimate beeing independent of a_1 , for any $x \in \overline{E}$ we have $|\partial P/\partial x_1(x)| \le 2 C' n^{r'} \|P\|_E$. From this we deduce $\|\partial P/\partial x_1\|_E \le C n^r \|P\|_E$

with $C = 2C'$ and $r = r'$. A similar process with variables $x_2,\dots,x_N$ instead of x_1 completes the proof.

References

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