CONSTRUCTIVE THEORY OF FUNCTIONS'87

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A MODIFIED BERNSTEIN-SCHOENBERG OPERATOR

T.N.T. Goodman and A. Sharma

1. Introduction

Take $n \geq 2$ and let $T = (t_j)_\alpha^\beta$, $\beta \geq \alpha + 2n + 1$, be an increasing sequence with multiplicities at most $n + 1$. We assume either $\alpha = -\infty$ or t_α has multiplicity $n + 1$ and similarly $\beta = \infty$ or t_β has multiplicity $n + 1$. Let $N_{n,j}$ be the normalised B-spline with knots $t_j, \ldots, t_{j+n+1}$, so that $\sum N_{n,j} = 1$. We shall denote by V_n^T the well-known variation-diminishing spline operator, introduced by Schoenberg [9], defined for any function f by

$$(1.1) \qquad V_n^T f(x) = \sum_{j=\alpha}^{\beta-n-1} f\left(\frac{1}{n}(t_{j+1} + \ldots + t_{j+n})\right) N_{n,j}(x) .$$

In this paper we introduce the operator U_n^T defined for any locally integrable function f by

$$(1.2) \qquad U_n^T f(x) = \sum_{j=\alpha}^{\beta-n-1} N_{n,j}(x) \int \frac{n-1}{t_{j+n} - t_{j+1}} N_{n-2,j+1}(t) f(t) dt ,$$

with the convention that if $t_{j+n} = t_{j+1}$, then the integral is replaced by $f(t_{j+1})$.

For $\beta = \alpha + 2n + 1$, $V_n^T f$ reduces to the Bernstein polynomial $B_n f$ of degree n, whereas $U_n^T f$ reduces to a polynomial similar to the modified Bernstein polynomial considered in [2], [3]. To reduce precisely to this polynomial we would need the degree of the B-splines inside the integral to be n. The reason we require these B-splines to have degree $n - 2$ is that this ensures that U_n^T, like V_n^T, reproduces linear functions, as is shown in §2.

In §2 we also show that U_n^T, like V_n^T, is variation diminishing in the sense of Schoenberg. This implies that if f is convex, then $U_n^T f$ is also convex, and in §3 we show that in this case $U_n^T f \geq f$ and adding extra knots to T decreases $U_n^T f$. Finally in §4 we derive bounds and asymptotic formulae for the error $U_n^T f - f$.

2. Shape Preserving Properties

Theorem 1. If f is linear, then $U_n^T f = f$.

Proof Clearly U_n^T reproduces constants. Now it is shown in [1] that

$$\int \frac{n-1}{t_{j+n} - t_{j+1}} N_{n-2,j+1}(t) t \, dt = \frac{1}{n} (t_{j+1} + \ldots + t_{j+n}) .$$

So if $f(x) = x$,

$$U_n^T f(x) = \sum_{j=\alpha}^{\beta-n-1} N_{n,j}(x) \frac{1}{n}(t_{j+1} + \ldots + t_{j+n})$$

$$= V_n^T f(x) = x. \qquad \square$$

Now given a sequence of numbers (a_j), finite or infinite, we denote by $S^-(a_j)$ the number of strict sign changes in the sequence. Given a function f on the real line we denote by $v(f)$ the number of strict sign changes of f, i.e. the supremum of $S^-(f(x_j))$ taken over all increasing sequences (x_j).

<u>Theorem 2.</u> For any locally integrable function f,

$$v(U_n^T f) \le v(f) .$$

<u>Proof</u> By the variation-diminishing properties of B-splines we have

$$v(U_n^T f) \le S^-(\int N_{n-2,j+1}(t)f(t)dt) , \tag{2.1}$$

with the previous convention that the integral is replaced by $f(t_{j+1})$ when $t_{j+n} = t_{j+1}$. Now suppose $\hat{T} = (\hat{t}_j)_\alpha^{\beta+1}$ is a new sequence formed by adding an extra element $\hat{t}_\ell$ to T. Thus $\hat{t}_j = t_j$ for $j < \ell$ and $\hat{t}_j = t_{j-1}$ for $j > \ell$. We assume $\hat{t}_\ell$ has multiplicity $\le n - 1$. We let $\hat{N}_{n-2,j}$ denote the corresponding B-splines and claim that

$$S^-(\int N_{n-2,j+1}(t)f(t)dt) \le S^-(\int \hat{N}_{n-2,j+1}(t)f(t)dt) , \tag{2.2}$$

with the previous convention.

If $t_{j+1} < t_{j+n}$, then by the positivity and basis property of B-splines, there are numbers $\alpha_{j+1} \ge 0$, $\beta_{j+1} \ge 0$ with

$$N_{n-2,j+1} = \alpha_{j+1} \hat{N}_{n-2,j+1} + \beta_{j+1} \hat{N}_{n-2,j+2} .$$

If $t_{j+1} = t_{j+n}$, then by convention

$$\int N_{n-2,j+1}(t)f(t)dt = f(t_{j+1}) = \int \hat{N}_{n-2,k}(t)f(t)dt ,$$

where $k = j + 1$ if $t_{j+1} < \hat{t}_\ell$ and $k = j + 2$ if $t_{j+1} > \hat{t}_\ell$. So for $j = \alpha, \ldots, \beta - n - 1$, there are numbers $\alpha_{j+1} \ge 0$, $\beta_{j+1} \ge 0$ with

$$\int N_{n-2,j+1}(t)f(t)dt = \alpha_{j+1} \int \hat{N}_{n-2,j+1}(t)f(t)dt + \beta_{j+1} \int \hat{N}_{n-2,j+2}(t)f(t)dt . \tag{2.3}$$

Since a one-banded positive matrix is totally positive, the variation diminishing property of totally positive matrices [7] shows that (2.3) implies (2.2). (This can also be shown without using total positivity as in Lemma 3 of [4].)

Now repeatedly add elements to T as above to form a sequence $\tilde{T}$ with an element of multiplicity $\ge n - 1$ at each point where f changes sign. We let $\tilde{N}_{n-2,j}$ denote the corresponding B-splines. If $\tilde{t}_k$, $\tilde{t}_\ell$ are consecutive points where f changes sign and $\tilde{t}_{k-n-2} = \ldots = \tilde{t}_k < \tilde{t}_\ell = \ldots = \tilde{t}_{\ell+n-2}$, then $\int \tilde{N}_{n-2,j}(t)f(t)dt$ has the same sign for $j = k-n-2,\ldots,\ell-1$. It follows that

$$S^-(\int \tilde{N}_{n-2,j+1}(t)f(t)dt) \le v(f) . \tag{2.4}$$

By repeated application of (2.2) we have

$$S^-(\int N_{n-2,j+1}(t)f(t)dt) \le S^-(\int \tilde{\tilde{N}}_{n-2,j+1}(t)f(t)dt) . \tag{2.5}$$

Combining (2.1), (2.5) and (2.4) gives the result. □

From theorems 1 and 2 it immediately follows that the number of times that the graph of $U_n^T f$ crosses a given straight line is no more than the number of times the graph of f crosses the line. In particular if f is convex, then $U_n^T f$ is convex.

3. Action on Convex Functions

Theorem 3. If f is convex, then $U_n^T f \ge V_n^T f$.

Proof Since f is convex and $(n-1)N_{n-2,j+1}/(t_{j+n} - t_{j+1})$ is positive with integral one, we have

$$f\Big(\int \frac{n-1}{t_{j+n}-t_{j+1}} N_{n-2,j+1}(t)\, t\, dt\Big) \le \int \frac{n-1}{t_{j+n}-t_{j+1}} N_{n-2,j+1}(t)f(t)dt . \tag{3.1}$$

From (1.2) and (3.1),

$$U_n^T f \ge \sum_{j=\alpha}^{\beta-n-1} N_{n,j}(x) f\Big(\int \frac{n-1}{t_{j+n}-t_{j+1}} N_{n-2,j+1}(t)\, t\, dt\Big)$$

$$= \sum_{j=\alpha}^{\beta-n-1} N_{n,j}(x) f\, \frac{1}{n}(t_{j+1} + \ldots + t_{j+n}) = V_n^T f(x) . \qquad \square$$

Since for convex f, $V_n^T f \ge f$ [6], we have

Corollary 1 If f is convex, then $U_n^T f \ge f$.

Theorem 4. If f is convex and T is a subsequence of S, then

$$U_n^S f \le U_n^T f .$$

Proof Clearly it is sufficient to prove the result for the case when S has one more element than T. Let $S = (\hat{t}_j)_\alpha^{\beta+1}$, where $\hat{t}_j = t_j$ for $j < \ell$ and $\hat{t}_j = t_{j-1}$ for $j > \ell$. Let $\hat{N}_{n,j}$, $\hat{N}_{n-2,j}$ denote the corresponding B-splines for S.

By the positivity and basis property of B-splines, there are numbers $\alpha_j \ge 0$, $\beta_j \ge 0$ such that

$$N_{n,j} = \alpha_j \hat{N}_{n,j} + \beta_j \hat{N}_{n,j+1}, \quad j = \alpha,\ldots, \beta - n - 1 \tag{3.2}$$

By (1.2) and (3.2),

$$U_n^T f(x) = \sum_{j=\alpha}^{\beta-n-1} \{\alpha_j \hat{N}_{n,j} + \beta_j \hat{N}_{n,j+1}\} \int \frac{n-1}{t_{j+n}-t_{j+1}} N_{n-2,j+1}(t)f(t)\, dt$$

$$= \sum_{j=\alpha}^{\beta-n} \hat{N}_{n,j}(x) \int (n-1) \{\alpha_j \frac{N_{n-2,j+1}(t)}{t_{j+n}-t_{j+1}} + \beta_{j-1} \frac{N_{n-2,j}(t)}{t_{j+n-1}-t_j}\} f(t)\, dt,$$

where $\alpha_{\beta-n} = \beta_{\alpha-1} = 0$. Thus

$$U_n^T f(x) - U_n^S f(x) = \sum_{j=\alpha}^{\beta-n} \hat{N}_{n,j}(x) \int Q_j(t) f(t)\, dt , \tag{3.3}$$

where

$$Q_j = \alpha_j \frac{(n-1)N_{n-2,j+1}}{t_{j+n} - t_{j+1}} + \beta_{j-1} \frac{(n-1)N_{n-2,j}}{t_{j+n-1} - t_j} - \frac{(n-1)\hat{N}_{n-2,j+1}}{\hat{t}_{j+n} - \hat{t}_{j+1}} \tag{3.4}$$

and as before a term is regarded as a δ-function if its denominator vanishes.

Now by (3.3) and Theorem 1, if ℓ is linear then

$$0 \equiv \sum_{j=\alpha}^{\beta-n} \hat{N}_{n,j}(x) \int Q_j(t) \ell(t)\, dt$$

and since the B-splines are linearly independent,

$$\int Q_j(t) \ell(t)\, dt = 0, \; j = \alpha, \ldots, \beta-n. \tag{3.5}$$

Defining F_j, $\alpha \leq j \leq \beta - n$, by

$$F_j(x) = \int_{\hat{t}_j}^{x} (x - t) Q_j(t)\, dt,$$

we see from (3.4) and (3.5) that F_j has support in $[\hat{t}_j, \hat{t}_{j+n+1}]$ and

$$F_j'' = Q_j \tag{3.6}$$

(in the sense of distributions if necessary). Thus F_j is a spline function of degree n with support in $[\hat{t}_j, \hat{t}_{j+n+1}]$ and knots at $\hat{t}_j, \ldots, \hat{t}_{j+n+1}$, and so $F = \lambda \hat{N}_{n,j}$ for some λ. Since $Q_j(x)$ is positive for x sufficiently close to $\hat{t}_j$, we have $\lambda \geq 0$.

Now applying (3.6) and approximating f by its Bernstein polynomials we have

$$\int Q_j(t) f(t) dt = \lim_{n\to\infty} \int F_j''(t) B_n f(t) dt$$

$$= \lim_{n\to\infty} \int F_j(t) (B_n f)''(t) dt \geq 0,$$

since F_j is positive and $B_n f$ is convex. Applying this inequality to (3.3) then gives $U_n^T f - U_n^S f \geq 0$ as desired. □

4. Convergence

Various results can be proved about the convergence of $U_n^T f$ to f as $n \to \infty$ and/or the mesh size $\to 0$, using techniques such as those in [8], [2] and [5]. Here we shall prove some of these.

We have seen that U_n^T reproduces linear functions. We now consider the error when U_n^T is applied to quadratic functions and set

$$E_n^T(x) := U_n^T(\cdot - x)^2(x) = (U_n^T g - g)(x), \quad g(t) = t^2.$$

It is known [1] that

$$(4.1) \qquad \int \frac{n-1}{t_{j+n}-t_{j+1}} N_{n-2,j+1}(t) t^2 \, dt = \sum_{j+1 \le k \le \ell \le j+n} t_k t_\ell / \binom{n+1}{2}.$$

Moreover it follows from Marsden's identity [8] that

$$(4.2) \qquad x^2 = \sum_{j=\alpha}^{\beta-n-1} N_{n,j}(x) \sum_{j+1 \le k < \ell \le j+n} t_k t_\ell / \binom{n}{2} .$$

Combining(1.2), (4.1) and (4.2) and simplifying gives

$$(4.3) \qquad E_n^T(x) = \sum_{j=\alpha}^{\beta-n-1} N_{n,j}(x) \frac{2}{(n+1)n(n-1)} \sum_{j+1 \le k < \ell \le j+n} (t_k - t_\ell)^2 .$$

It will be useful to compare (4.3) with the corresponding expression for the operator V_n^T. If we write

$$e_n^T(x) := V_n^T(\cdot - x)^2(x) = (V_n^T g - g)(x), \quad g(t) = t^2,$$

then it is known [8] that

$$e_n^T(x) = \sum_{j=\alpha}^{\beta-n-1} N_{n,j}(x) \frac{1}{n^2(n-1)} \sum_{j+1 \le k < \ell \le j+n} (t_k - t_\ell)^2 .$$

Thus

$$(4.4) \qquad E_n^T = \frac{2n}{n+1} e_n^T .$$

For any continuous function f on an interval I we define its modulus of continuity for $h > 0$ by

$$\omega(f;h) := \sup_{\substack{x,y \in I \\ |x-y| \le h}} |f(x) - f(y)| .$$

From (4.3) we can estimate the error in $U_n^T f$ for any continuous function f using the following result of Shisha and Mond [10].

__Theorem A.__ If L is a positive linear operator from $C([a,b])$ to itself which reproduces constant functions, then for any f in $C([a,b])$,

$$\sup_{x \in [a,b]} |Lf(x) - f(x)| \le 2\omega(f;\mu)$$

where

$$\mu^2 = \sup_{x \in [a,b]} L(\cdot - x)^2(x) .$$

__Theorem 5.__ For a continuous function f and $t_i \le x < t_{i+1}$ we have

$$|U_n^T f(x) - f(x)| \le 2\omega(f|I_n ; \mu_n) ,$$

where $I_n = [t_{i-n+1}, t_{i+n}]$ and

$$\mu_n = \frac{1}{\sqrt{n+1}} \max_{i-n \le j \le i} |t_{j+n} - t_{j+1}| .$$

Proof The value of $U_n^T f$ is not altered if we add extra knots at t_{i-n+1} and t_{i+n} until they both have multiplicity $n + 1$. The result then follows from (4.3) and Theorem A. □

We note that if T is restricted to a fixed interval $[a,b]$, then $U_n^T f \to f$ uniformly as $n \to \infty$. In any case if $\sqrt{n}\max(t_{j+1} - t_j) \to 0$ as $n \to \infty$, then $U_n^T f \to f$ uniformly on any interval on which f is uniformly continuous. The rate of convergence can be improved if we assume $f \in C^2$.

Theorem 6. If $f \in C^2$ and if $t_i \le x < t_{i+1}$, then

$$(4.5) \qquad |U_n^T f(x) - f(x)| \le \tfrac{1}{2}\mu_n^2 \max\{|f''(t)| : t_{i-n+1} \le t \le t_{i+n}\},$$

where μ_n is as in Theorem 5.

Proof By Taylor's Theorem for $t_{i-n+1} \le t \le t_{i+n}$,

$$(4.6) \qquad f(t) = f(x) + (t - x)f'(x) + g(t)$$

where

$$(4.7) \qquad |g(t)| \le \tfrac{1}{2}K(t - x)^2, \ K = \max\{|f''(s)| : t_{i-n+1} \le s \le t_{i+n}\},$$

As before we may assume that T has knots of multiplicity $n + 1$ at t_{i-n+1} and t_{i+n} and applying U_n^T to (4.6) on $[t_{i-n+1}, t_{i+n}]$ and recalling Theorem 1 gives

$$|U_n^T f(x) - f(x)| = |U_n^T g(x)| \le \tfrac{1}{2}KE_n^T(x)$$

by (4.7). The result then follows from (4.3). □

Finally we shall follow [5] in deriving an asymptotic formula for the error $U_n^T f(x) - f(x)$ as $n \to \infty$. We have seen above how we can restrict attention to a finite interval by adding appropriate knots and we now assume T is on a fixed finite interval which, without loss of generality, we take to be $[0,1]$. For simplicity we take $\alpha = -n$, $\beta = N + n$, so that $T = (t_j)_{-n}^{N+n}$ with

$$t_{-n} = \dots = t_0 = 0, \ t_N = \dots = t_{N+n} = 1.$$

We would expect an asymptotic formula to exist only if T has some form of limiting distribution as $n \to \infty$. We describe the distribution of T by the function g_T defined by

$$g_T(t) = \begin{cases} 0 , & t \le 0, \\ t_i, & \frac{i-1}{n} < t \le \frac{i}{n}, \ i = 1,\dots,N, \\ 1 , & t \ge N/n. \end{cases}$$

We shall assume that as $n \to \infty$, $N/n \to \lambda$, $0 \leq \lambda < \infty$, and g_T converges almost everywhere to a function g. Then g is an increasing function almost everywhere and satisfies $g(t) = 0$ for $t < 0$ and $g(t) = 1$ for $t > \lambda$. We shall need the further requirement that

$$0 \leq x + 1 < y \leq \lambda + 1 \Rightarrow g(x) < g(y)$$

which can be regarded as the limiting case of the condition that T has multiplicities at most $n + 1$.

<u>Theorem 7.</u> If f is bounded and has a second derivative at the point x, then

$$(4.8) \qquad \lim_{n\to\infty} n\left(U_n^T f(x) - f(x)\right) = f''(x)e(x)$$

where

$$e(x) = \int_{a(x)}^{a(x)+1} g^2(t)dt - x^2$$

and $a(x)$ is the unique number in $[-1,\lambda]$ satisfying

$$\int_{a(x)}^{a(x)+1} g(t)dt = x \ .$$

<u>Proof</u> In Theorem 2 of [5] it is shown that $\lim_{n\to\infty} ne_n^T(x) = e(x)$ and so (4.4) gives

$$(4.9) \qquad \lim_{n\to\infty} n\, E_n^T(x) = 2e(x) \ .$$

If we write, for $0 \leq t \leq 1$,

$$(4.10) \qquad f(t) = f(x) + (t - x)f'(x) + \tfrac{1}{2}(t - x)^2 f''(x) + R(t) \ ,$$

then it is easily shown (see the proof of Theorem 1 in [5]) that for any $\epsilon > 0$ there is a constant C, depending on ϵ, with

$$(4.11) \qquad |R(t)| < \epsilon(t - x)^2 + C(t - x)^4 \ , \ 0 \leq t \leq 1 \ .$$

By Theorem 4,

$$(4.12) \qquad U_n^T(. - x)^4 \leq \tilde{B}_n(. - x)^4$$

and a direct calculation shows that

$$(4.13) \qquad \tilde{B}_n(. - x)^4 = 0(1/n^2) \ .$$

So by (4.11), (4.12) and (4.13),

$$|U_n^T R(x)| \leq \epsilon E_n^T(x) + 0(1/n^2)$$

and recalling (4.9) we have

$$(4.14) \qquad \lim_{n\to\infty} n\, U_n^T R(x) = 0 \ .$$

But by (4.10) ,

$$U_n^T f(x) = f(x) + \frac{1}{2} E_n^T(x) f''(x) + U_n^T R(x)$$

and applying (4.9) and (4.14) gives (4.8) . □

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