

UNCONDITIONAL BASES IN TRANSLATION AND DILATION

INVARIANT FUNCTION SPACES ON \mathbb{R}^n

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1. Introduction. The existence of an unconditional basis is known for many function spaces as well as explicit constructions for such bases. It is a classical result that the Haar system is a basis of $L^p(\mathbb{R}^n)$ for $1 < p < \infty$. For spaces with intrinsic smoothness more subtle constructions are needed. As examples we mention bases in the Besov spaces on \mathbb{R}^n for a certain range of parameters due to Triebel ([T3] and [T1], p.86 for further references), bases for the real Hardy space H^1 ([C] and [W]) and the work of Ciesielski (e.g. [CI]) on spline bases for function spaces on compact manifolds. Recently a new type of bases for function spaces on \mathbb{R}^n was discovered which reconciles the simplicity of the Haar system and the need for smoothness. In these so-called wavelet bases of P.-G. Lemarié and Y. Meyer [LM] all elements are translates and dilates of a finite number of Schwartz functions and thus they are especially well adapted to translation and dilation invariant function spaces on \mathbb{R}^n . In fact, they proved that a wavelet basis is an unconditional basis for all classical function spaces on \mathbb{R}^n , among them L^p , the Sobolev spaces, the Besov spaces, the real Hardy space H^1 etc. For the proof they used deep results on the boundedness of Calderon-Zygmund operators on these spaces and thus it was not clear how to decide if a wavelet basis is a basis for other spaces on \mathbb{R}^n .

In this article we want to show that the real power of the wavelet bases is exhibited only in the frame of the coorbit spaces of the $ax+b$ -group. These build a general scale of translation and dilation invariant function spaces on \mathbb{R}^n which contains all the above-mentioned classical function spaces. The methods and techniques that were developed in [FG1] and [FG2] to describe these spaces and to derive their atomic decompositions then allow a rather direct approach to the basis problem. Relying on assertions of [FG2] we shall show that a wavelet basis is an

unconditional basis of every coorbit space in which the Schwartz class is dense. This result contains the theorem of [LM] and the method of coorbit spaces provides a new proof of it.

2. Wavelet bases in $L^2(\mathbb{R}^n)$

A wavelet basis of $L^2(\mathbb{R}^n)$ is a complete orthonormal system whose elements are of the special form $2^{jn/2} g_r(2^j x - k)$, $k \in \mathbb{Z}^n$, $j \in \mathbb{Z}$, $r=1, \dots, 2^n-1$, and satisfy smoothness and decay conditions. Writing $L_x f(y) = f(y-x)$ for the usual translations and $D_t f(y) = t^{-n/2} f(y/t)$ for the L^2 -isometric dilations, we see that a wavelet basis is contained in the orbit of a finite number of elements g_r under the group of unitary operators $\{ L_x D_t, x \in \mathbb{R}^n, t > 0 \}$. The rather surprising existence of such bases was proved by P.-G. Lemarié and Y. Meyer [LM] in 1985.

Theorem A. One can construct 2^n-1 functions $g_r \in L^2(\mathbb{R}^n)$, $r=1, \dots, 2^n-1$, satisfying prescribed smoothness conditions such that the set $\{ D_{2^j} L_k g_r, k \in \mathbb{Z}^n, j \in \mathbb{Z}, r=1, \dots, 2^n-1 \}$ is a wavelet basis of $L^2(\mathbb{R}^n)$. As regularity conditions are admitted for example

- (1) smoothness of order m ($m \in \mathbb{N}$ fix)

$$\sup_{x \in \mathbb{R}^n} (1 + |x|)^{n+m} |D^\alpha g_r(x)| \leq C \quad (2.1)$$

for $|\alpha| \leq m$, and $r = 1, \dots, 2^n-1$.

- (2) infinite smoothness: $g_r \in \mathcal{S}'_0$,

where even $\text{supp } \hat{g}_r$ may be compact for all r . (\mathcal{S}'_0 denotes the space of all Schwartz functions all whose moments vanish, \mathcal{S}'_0 its dual space which may be identified to the tempered distributions modulo polynomials)

We shall not give the details of the construction, but rely exclusively on the fact that the elements of the basis are translates and dilates of a finite number of functions which satisfy smoothness and decay properties.

For systematic algorithms to produce wavelet bases with the help of the so-called multiscale analysis and for applications in signal analysis, quantum field theory, numerical analysis or the theory of Calderon-Zygmund operators the reader is referred to the recent works of Y. Meyer and his group, in particular to [JLM], [M], and a note of the author [G1].

3. Coorbit spaces under the $ax+b$ -group

In [FG1] and [FG2] a general theory of Banach spaces related to

integrable group representations which are called *coorbit spaces* is developed. Given an irreducible, unitary, continuous representation π of a locally compact group \mathcal{G} on a Hilbert space \mathcal{H} , which is at least integrable, one can associate to any solid, translation-invariant Banach space of functions Y on the group \mathcal{G} a Banach space $\mathcal{C}\alpha Y$ which "lives" on the same level as \mathcal{H} . The interest in this construction originates from the fact that several classical families of function spaces (on \mathbb{R}^n and on the unit disc resp.) arise as coorbits under certain group representations. Moreover, there exists now a detailed theory of atomic decompositions for these spaces which contains many known results and provides new ones. For the general theory we refer to [FG1] and [FG2].

In this section we give some details for the coorbits under the $ax+b$ -group to be used in the fourth section. The assertions in this section are consequences of the general theorems in [FG2]. The following discussion of the hypotheses shows that they are actually applicable.

Let $\mathcal{G} = \mathbb{R}^n \times \mathbb{R}^+$ be the n -dimensional $ax+b$ -group with multiplication

$$(x,t)(y,v) = (x+ty, tv), \quad x,y \in \mathbb{R}^n, t,v \in \mathbb{R}^+ \quad (3.1)$$

The left Haar measure on \mathcal{G} is $dxdt/t^{n+1}$, the Haar module is $\Delta(x,t) = t^{-n}$. We shall often write \underline{x} for the group element $(x,t) \in \mathcal{G}$. \mathcal{G} possesses a canonical unitary representation π on $L^2(\mathbb{R}^n)$ which is defined by

$$\pi(\underline{x})f(y) = \pi(x,t)f(y) = L_x D_t f(y) = D_t L_{x/t} f(y) = t^{-n/2} f\left(\frac{y-x}{t}\right) \quad (3.2)$$

π is an integrable, unitary representation of \mathcal{G} and possesses even rapidly decreasing representation coefficients $\langle L_x D_t g, g \rangle$, i.e. there exist $g \in L^2(\mathbb{R}^n)$ such that

$$\int |\langle L_x D_t g, g \rangle| (1+|x|)^r (t^s + t^{-s'}) dxdt/t^{n+1} < \infty \quad (3.3)$$

for all $r \in \mathbb{N}$, $s, s' \in \mathbb{R}$, e.g. every $g \in \mathcal{S}_0$ satisfies (3.3).

Let Y be a Banach space of measurable functions on \mathcal{G} which satisfies:

(3.4) Y is continuously embedded in $L^1_{loc}(\mathcal{G})$

(3.5) Y is invariant under left and right translations $L_{\underline{x}}, R_{\underline{x}}$ of \mathcal{G} .

(3.6) Y is solid, i.e., $H \in Y$, and $|F(\underline{x})| \leq |H(\underline{x})|$ a.e. imply $F \in Y$ and $\|F\|_Y \leq \|H\|_Y$

(3.7) The norm of the right translation operator $w(x,t) := \|R_{(x,t)}\|_Y$ (where $R_{\underline{x}} F(\underline{z}) = F(\underline{z}\underline{x})$ for functions F on \mathcal{G}) is bounded by $(1+|x|)^r (t^s + t^{-s'})$ for some $r, s, s' \geq 0$.

Finally fix a radial Schwartz function $g \in \mathcal{S}_0$ whose Fourier transform \hat{g} has compact support not containing the origin 0. We can now define the

coorbit of Y under the representation π .

Definition 1. $\mathcal{E}oY = \{ f \in \mathcal{S}'_0 \mid V_g(f)(x,t) = \langle L_x D_t g, f \rangle \in Y \}$

with the norm $\| f \|_{\mathcal{E}oY} = \| V_g(f) \|_Y$

Thus a distribution f belongs to $\mathcal{E}oY$ if and only if the extended representation coefficient $\langle L_x D_t g, f \rangle$, which is a continuous function of (x,t) on \mathcal{G} , is in Y . The bracket $\langle \dots \rangle$ denotes the pairing between test functions and distributions. We collect some properties of $\mathcal{E}oY$.

Proposition. ([FG1], Thm. 5.2) Assume that (3.4) - (3.7) are true. Then

- (i) $\mathcal{E}oY$ is a Banach space of distributions which is invariant under translations L_x , $x \in \mathbb{R}^n$, and dilations D_t , $t > 0$.
- (ii) The definition of $\mathcal{E}oY$ is independent of $g \in \mathcal{S}_0$ in the sense that different functions g yield equivalent norms of $\mathcal{E}oY$.
- (iii) One has the embeddings: $\mathcal{S}_0 \rightarrow \mathcal{E}oY \rightarrow \mathcal{S}'_0$. If the measurable functions of compact support are dense in Y , \mathcal{S}_0 is dense in $\mathcal{E}oY$.

Remarks. (i) Condition (3.7) is necessary only to stay within the tempered distributions. It would be no difficulty to extend the theory to the realm of ultra-distributions. Moreover, the set of admissible vectors g in (ii) is much larger than \mathcal{S}_0 and can be described by condition (3.3) the parameters r, s, s' depending on the associated weight w of Y (3.7). Cf. [FG1, 2].

(ii) The attentive reader may have observed that the representation $(x,t) \rightarrow L_x D_t$ is not irreducible and thus the results of [FG1, 2] might not be applicable. We escaped this difficulty assuming that g is radial. Then the orthogonality relations and the reproducing formula still hold and this was all that we needed in [FG1, 2]. Another (equivalent) way is to consider the larger group $\mathbb{R}^n \times (\mathbb{R}^+ \times O(n))$ with its natural representation by translations, dilations and rotations on $L^2(\mathbb{R}^n)$. This representation is irreducible and our theory works again. In this case the function g need not be radial.

The most prominent members in this scale of coorbits are the *spaces of Besov-Triebel-Lizorkin type*. They are originally defined by decomposition properties of their Fourier transform: let $g \in \mathcal{S}$ be an admissible function in the sense of (3.3) and assume that $\sum_{j=-\infty}^{\infty} \hat{g}(2^j x) = 1 \quad \forall x \neq 0$. Then the homogeneous Besov space $B_{p,q}^s$ for $s \in \mathbb{R}$, $0 < p, q \leq \infty$ is defined by

$$B_{p,q}^s = \{ f \in \mathcal{S}'_0 \mid \left[\sum_{j=-\infty}^{\infty} \| \mathcal{F}^{-1}(\hat{g}(2^j \cdot) \hat{f}) \|_p^q 2^{qjs} \right]^{1/q} < \infty \} \quad (3.8)$$

and the Triebel-Lizorkin-space $F_{p,q}^s$, $s \in \mathbb{R}$, $0 < q \leq \infty$, $0 < p < \infty$ by

$$F_{p,q}^s = \{f \in \mathcal{Y}' \mid \| \left[\sum_{j=-\infty}^{\infty} |\mathcal{F}^{-1}(\hat{g}(2^j \cdot) \hat{f})(x)|^q 2^{qjs} \right]^{1/q} L^p(dx) \| < \infty \}. \quad (3.9)$$

All the classical function spaces on \mathbb{R}^n (L^p , the Sobolev spaces, the Hölder-Zygmund spaces, the real Hardy spaces etc.) are contained in these two scales. For these well-known facts we refer to the book of Triebel [T1]. Next we identify these spaces as coorbits under the representation $(x,t) \rightarrow L_{x,t} D_t$, where we restrict to the case of Banach spaces, i.e. to the parameters $s \in \mathbb{R}$, $1 \leq p, q \leq \infty$ (and $p, q < \infty$ for the F-spaces).

Theorem B. With the notation $L_{p,q}^s = \{ F(x,t) \text{ measurable on } \mathcal{G} \text{ such that}$

$$\left[\int_{\mathbb{R}^+} \left[\int_{\mathbb{R}^n} |F(x,t)|^p dx \right]^{q/p} t^{-qs} dx dt / t^{n+1} \right]^{1/q} < \infty \} \text{ and } T_{p,q}^s = \{ F(x,t)$$

$$\text{with } \left[\int_{\mathbb{R}^n} \left[\int_{|y-x| \leq t} |F(y,t)|^q t^{-qs} dy dt / t^{n+1} \right]^{p/q} dx \right]^{1/p} < \infty \}$$

we have

$$B_{p,q}^s = \mathcal{C}_\alpha L_{p,q}^{s+n/2-n/q} \quad \text{and} \quad F_{p,q}^s = \mathcal{C}_\alpha T_{p,q}^{s+n/2} \quad (3.10)$$

This is a mere rephrasing in our terminology of a theorem of Triebel on equivalent norms for the B- and F-spaces, cf. [T2], 2.3. Cor.10 and 2.2. Remark 13. But now the basic properties of these spaces follow from [FG1,2] and the properties of the function spaces on the group \mathcal{G} without recourse to their specific structure and Fourier analysis. This point will be pursued further in [G2]. The $T_{p,q}^s$, the so-called *tent spaces*, were only recently introduced in [CMS] to study maximal inequalities from an abstract point of view.

Next we consider the atomic decompositions of the coorbit spaces. For that purpose one associates to a function space Y on \mathcal{G} a sequence space Y_d in the following way. For $\alpha > 0$ and $\beta > 1$ set $Q = [0, \alpha]^n \times [1, \beta]$ and $\underline{x}_{j,k} = (\alpha k \beta^j, \beta^j)$, $(j,k) \in \mathbb{Z} \times \mathbb{Z}^n$. Then the translates $\underline{x}_{j,k} Q$ form a partition of \mathcal{G} and one can give without problem the following

Definition 2.

$$Y_d(\alpha, \beta) = \{ (\lambda_{j,k})_{(j,k) \in \mathbb{Z} \times \mathbb{Z}^n} \mid H(y,t) := \sum_{j,k} \lambda_{j,k} L_{\underline{x}_{j,k}} 1_Q(y,t) \in Y \}$$

$$\text{with the norm } \| (\lambda_{j,k}) \mid Y_d(\alpha, \beta) \| := \| H \mid Y \|$$

Then Y_d is a Banach space of sequences whose properties are intimately related to those of the function space Y , e.g. $L^p(\mathcal{G})_d = \ell^p$.

The atomic decomposition for translation and dilation invariant function spaces can now be formulated as follows:

Theorem C. ([FG2], 6.1) Let Y be a function space on \mathcal{G} which satisfies the hypotheses of (3.4) - (3.7) and let $w(x,t)$ be the norm of right translations on Y .

(i) For any $g \in L^2(\mathbb{R}^n)$ satisfying the condition

$$\int_{\mathcal{G}} \left[\sup_{\substack{|\xi-x| \leq t \\ t/2 \leq \tau \leq t}} | \langle L_{\xi}^D g, g \rangle | \right] w(x,t) dx dt / t^{n+1} < \infty \quad (3.11)$$

there exist $\alpha_0 > 0$ and $\beta_0 > 1$ depending only on $g \in L^2(\mathbb{R}^n)$ with the following property: If $0 < \alpha \leq \alpha_0$ and $1 < \beta \leq \beta_0$ then any $f \in \mathcal{C}\alpha Y$ has a representation

$$f = \sum_{j,k} \lambda_{j,k} L_{\alpha\beta^j k}^D \beta^j g \quad (3.12)$$

with coefficients $\lambda_{j,k}$ in $Y_d(\alpha, \beta)$ depending linearly on f and satisfying the estimate

$$\| (\lambda_{j,k}) | Y_d(\alpha, \beta) \| \leq C_0 \| f | \mathcal{C}\alpha Y \| \quad (3.13)$$

(ii) Conversely, for any sequence $(\lambda_{j,k}) \in Y_d(\alpha, \beta)$ the element

$f := \sum_{j,k} \lambda_{j,k} L_{\alpha\beta^j k}^D \beta^j g$ is in $\mathcal{C}\alpha Y$ and one has

$$\| f | \mathcal{C}\alpha Y \| \leq C_1 \| (\lambda_{j,k}) | Y_d(\alpha, \beta) \| \quad (3.14)$$

The sums converge in the norm of $\mathcal{C}\alpha Y$ if the bounded functions of compact support are dense in Y and in the distributional sense otherwise. The constants C_0, C_1 depend only on the atom g , but not on α and β .

Remarks. (i) Instead of the regular sampling points $(\alpha\beta^j k, \beta^j)$ rather irregular families of points in \mathcal{G} are allowed as long as they are distributed sufficiently dense and not too concentrated. (Cf [FG2], §3. and Thm. 6.1).

(ii) The functions satisfying (3.11) are a dense subspace of $L^2(\mathbb{R}^n)$ (cf. [FG1] Lemma 6.1(i)) and it is easy to check that (3.11) holds for every $f \in \mathcal{S}_0$. This is all we shall use in the sequel.

From now on the atomic decompositions of the classical function spaces (i.e. $L^p(\mathbb{R}^n)$, the Sobolev-spaces, the Hölder-Zygmund spaces, the Hardy spaces etc.) are obtained in a straightforward manner. Since they are all contained in the scale of Besov-Triebel-Lizorkin spaces one has to calculate the sequence spaces associated to $L_{p,q}^s$ and $T_{p,q}^s$, which is a medium size calculation starting from Definition 2. For the explicit statements we refer to [G2] and to the articles of Frazier and Jawerth [FJ1,2] who have worked out a similar theory of decompositions of the

Besov-Triebel-Lizorkin spaces with completely different methods.

In the next section we shall need the following theorem on the "sampling values" of the extended representation coefficients:

Theorem D. Let $g \in \mathcal{S}_0$ be fixed. If $f \in \mathcal{C}\alpha Y$ then the coefficient sequence $c_{j,k} := \langle L_{\alpha k \beta^j}^D g, f \rangle$, $(j,k) \in \mathbb{Z} \times \mathbb{Z}^n$ is in $Y_d(\alpha, \beta)$ and satisfies

$$\| (c_{j,k})_{\mathbb{Z} \times \mathbb{Z}^n} \|_{Y_d(\alpha, \beta)} \leq C_2 \| f | \mathcal{C}\alpha Y \| \quad (3.15)$$

The constant C_2 depends only on g , not on α and β .

This is a variant of Theorem 5.8 of [FG2]. The hypothesis which is made there is satisfied as a consequence of (3.11).

4. Unconditional bases in coorbit spaces

The wavelet bases and the coorbit spaces fit together very well and in this section we show that the wavelet bases are unconditional bases of all these spaces. With the help of the atomic decomposition for these spaces and the general methods of [FG1,2] it is not difficult to derive these results.

Let us first recall that an unconditional basis of a Banach space B is a sequence of vectors $\{e_n\}_{n \in \mathbb{N}} \subseteq B$ with the following properties:

(4.1) The finite linear combinations of the e_n generate a dense subspace of B .

(4.2) There exists a constant $C > 0$ such that for any $n \in \mathbb{N}$ and arbitrary finite sequences $(a_i), (b_i), i=1, \dots, n$ with $|a_i| \leq |b_i|$ for $i=1, \dots, n$ one has the estimate:

$$\| \sum_{i=1}^n a_i e_i \| \leq C \| \sum_{i=1}^n b_i e_i \|$$

Another way of expressing these conditions is to say that the e_n are a Schauder basis such that the series expansions $\sum_{n=1}^{\infty} c_n e_n \in B$ together with all its rearrangements converge in B .

Theorem. Let Y be a function space on \mathcal{F} for which the assumptions (3.4) -(3.7) are fulfilled and such that the bounded, measurable functions of compact support are dense in Y . Then every wavelet basis of infinite smoothness $\{ L_{k 2^j}^D g_r, k \in \mathbb{Z}^n, j \in \mathbb{Z}, r=1, \dots, 2^n-1 \}$ is an unconditional basis of $\mathcal{C}\alpha Y$ and every $f \in \mathcal{C}\alpha Y$ has the representation

$$f = \sum_{j,k,r} \langle L_{2^j k}^D g_r, f \rangle L_{2^j k}^D g_r \quad (4.3)$$

The coefficient mapping $f \rightarrow (\langle L_{2^j k}^D g_r, f \rangle)_{(j,k) \in \mathbb{Z} \times \mathbb{Z}^n, r=1, \dots, 2^n-1}$ is an

isomorphism from $\mathcal{E}\alpha Y$ onto the sequence space $\sum_{r=1}^{2^n-1} Y_d(1,2)$.

Modulo the calculation of the sequence spaces we obtain with Thm B the following corollary which contains the previously known results of [LM]:

Corollary. Every wavelet basis of infinite smoothness is an unconditional basis simultaneously for all spaces of Besov-Triebel-Lizorkin-type $\dot{B}_{p,q}^s$ and $\dot{F}_{p,q}^s$. More precisely, let $f \in \mathcal{S}'_0$ be a distribution with the expansion (4.3). Then $f \in \dot{B}_{p,q}^s$, $s \in \mathbb{R}$, $1 \leq p, q < \infty$ if and only if

$$\|f|_{\dot{B}_{p,q}^s}\| \cong \left(\sum_{j \in \mathbb{Z}} \left[\sum_{\substack{k \in \mathbb{Z}^n \\ r=1, \dots, 2^n-1}} |\lambda_{j,k,r}|^p \right]^{q/p} 2^{-jq(s+n/2-n/p)} \right)^{1/q} < \infty$$

and $f \in \dot{F}_{p,q}^s$, $s \in \mathbb{R}$, $1 \leq p, q < \infty$ if and only if $\|f|_{\dot{F}_{p,q}^s}\| \cong$

$$\cong \left\| \left[\sum_{j,k,r} |\lambda_{j,k,r}|^q 2^{-jq(s+n/2)} \right]_{[2^j k, 2^j(k+1)]} (x) \right\|^{1/q} \|L^p(dx)\| < \infty.$$

Remark. Analogous results hold true for weighted versions of the B- and F-spaces if the weight function grows at most polynomially.

As another consequence we obtain the following

Corollary. If the functions of compact support are not dense in Y then the system $\{L_{2^j k}^D 2^{j \alpha_r} g_r\}$ is a weak basis of $\mathcal{E}\alpha Y$, i.e. every element $f \in \mathcal{E}\alpha Y$ has still an expansion (4.3) with coefficients $\langle L_{2^j k}^D g_r, f \rangle$ in $\sum_{r=1}^{2^n-1} Y_d(1,2)$, but the sum is only w^* -convergent in \mathcal{S}'_0 .

Proof of the Theorem. For the proof we shorten the notation and write $I := \mathbb{Z} \times \mathbb{Z}^n$ for the index set, $\pi(\underline{x})$ for the representation $L_{\underline{x}}^D g_t$ and $\underline{x}_i := (k2^j, 2^j)$ or $(\underline{y}_i) := (\alpha k \beta^j, \beta^j)$ for the sampling points in \mathcal{S} . First of all we note that we may apply Thm.C and Thm.D without further comment because the basic functions g_r are in \mathcal{S}'_0 and thus satisfy the conditions (3.3) and (3.11).

We show directly the conditions (4.1) and (4.2).

(a) Take two finite sequences $(a_i), (b_i)$, $i \in F$, $F \subseteq I$ being a finite set, such that $|a_i| \leq |b_i|$ for $i \in F$ holds, and consider the elements $f = \sum_{i \in F} a_i \pi(\underline{x}_i) g_r$ and $h = \sum_{i \in F} b_i \pi(\underline{x}_i) g_r$ for some $r=1, \dots, 2^n-1$. Then we estimate

$$\|f|_{\mathcal{E}\alpha Y}\| \leq C_1 \|(a_i)|_{Y_d(\alpha, \beta)}\| \leq C_1 \|(b_i)|_{Y_d(\alpha, \beta)}\| \leq C_1 C_2 \|h|_{\mathcal{E}\alpha Y}\| \quad (4.4)$$

where the first inequality follows from Thm.C(ii), the second one from

the definition of Y_d . The third one is a consequence of the orthogonality of the wavelet basis ($\Rightarrow a_1 = \langle \pi(x_1)g_r, f \rangle$) and Thm.D. It is clear that it suffices to have this estimate separately for each $r=1, \dots, 2^n-1$.

(b) We want to show that the finite linear combinations of the $D_{2^j} L_k g_r$ are dense in $\mathcal{C}aY$. For that purpose we start with an $\varepsilon > 0$ and an $f \in \mathcal{C}aY$, which has a decomposition $f = \sum_{i \in I} \lambda_i \pi(y_1)g$ for some fixed $g \in \mathcal{S}_0$ and $\alpha, \beta > 0$ sufficiently small (Thm.C(i)). Since the bounded functions of compact support are dense in Y , the finite sequences are dense in $Y_d(\alpha, \beta)$. Thus there is a finite set $F \subseteq I$ such that $\| (\lambda_i)_{i \notin F} |_{Y_d(\alpha, \beta)} \| < \varepsilon$ and consequently one has for the element

$$f_0 = \sum_{i \in F} \lambda_i \pi(y_1)g \in \mathcal{C}aY \cap L^2(\mathbb{R}^n) \quad (4.5)$$

$$\| f - f_0 |_{\mathcal{C}aY} \| \leq C_1 \varepsilon \quad (4.6)$$

by (3.14). $f_0 \in \mathcal{C}aY \cap L^2(\mathbb{R}^n)$ has the orthogonal expansion

$$f_0 = \sum_r \sum_{i \in I} \langle \pi(x_i)g_r, f_0 \rangle \pi(x_i)g_r \quad (4.7)$$

Thm.D then asserts that the coefficient sequence $c_{i,r} := \langle \pi(x_i)g_r, f \rangle$, $i \in I$ is in $Y_d(1,2)$ for each $r=1, \dots, 2^n-1$. The same argument as above yields the existence of finite sets $F_r \subseteq I$ such that

$$\| (c_{i,r})_{i \notin F_r} |_{Y_d(1,2)} \| < \varepsilon \quad \text{for all } r \quad (4.8)$$

Then the element

$$f_1 = \sum_r \sum_{i \in F_r} \langle \pi(x_i)g_r, f_0 \rangle \pi(x_i)g_r \quad (4.9)$$

is the desired approximation of f in $\mathcal{C}aY$ by finite linear combinations:

$$\begin{aligned} \| f - f_1 |_{\mathcal{C}aY} \| &\leq \| f - f_0 |_{\mathcal{C}aY} \| + \| f_0 - f_1 |_{\mathcal{C}aY} \| \leq \\ &\leq C_1 \varepsilon + \sum_r \| \sum_{i \notin F_r} \langle \pi(x_i)g_r, f_0 \rangle \pi(x_i)g_r |_{\mathcal{C}aY} \| \leq \\ &\leq C_1 \varepsilon + (2^n - 1)C_1 \varepsilon = 2^n C_1 \varepsilon \end{aligned} \quad (4.10)$$

where we have used Thm.C(ii) and (4.7).

Altogether we have proved that the set $\{ \pi(x_i)g_r, i \in I, r=1, \dots, 2^n-1 \}$ is an unconditional basis of $\mathcal{C}aY$.

When one replaces f_1 by $f_2 = \sum_r \sum_{i \in F_r} \langle \pi(x_i)g_r, f \rangle \pi(x_i)g_r$ then one obtains from (4.5) with (3.14) and (3.15) $\| f_1 - f_2 |_{\mathcal{C}aY} \| \leq 2^n C_1^2 C_2 \varepsilon$. Thus f has the representation $f = \sum_{r,i} \langle \pi(x_i)g_r, f \rangle \pi(x_i)g_r$ where the partial sums converge to f in the $\mathcal{C}aY$ -norm. The rest is now clear.

Remark. We have treated only the wavelet basis of infinite smoothness because we wanted to demonstrate that they are unconditional bases for simultaneously for all coorbit spaces. The wavelets with smoothness m (1.1) are still unconditional bases for $\mathcal{W}oY$ as long as Y satisfies (3.7) for certain parameters r, s, s' depending on m . After having checked that the basis vectors g_r fulfill (3.11) the proof is the same.

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