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A PROBLEM ON BLASHKE PRODUCTS

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1. Introduction. Let $r_1,\dots,r_n$ be given positive integers and

$$\Omega_n=\{ (t_1,\dots,t_n)\in R^n : -1<t_1<\dots<t_n<1 \}.$$

The Blashke product with zeros $\bar{x}=(x_1,\dots,x_n)\in\Omega_n$ of multiplicities $r_1,\dots,r_n$ respectively is defined by

$$(1)\qquad B(\bar{x};t)=\prod_{i=1}^{n}[(t-x_i)/(1-tx_i)]^{r_i}.$$

We consider the problem on existence and uniqueness of Blashke product with fixed multiplicities of zeros, properly normalized by a constant c, so that the function $f(t)=B(\bar{x};t)/c$ and the real line to enclose prescribed areas in the intervals $[-1,x_1]$, $[x_1,x_2],\dots,[x_n,1]$.

2. Main result. The next auxiliary lemma is proved in [3].

Lemma. Let $\bar{x}=(x_1,\dots,x_n)\in\Omega_n$. Then the functions

$$u_i(t)=1/[(t-x_i)(1-tx_i)],\qquad i=1,\dots,n$$

form a Chebyshev system on the set $A:=(-1,1)\setminus\{x_1,\dots,x_n\}$.

We now formulate our result.

Theorem. Let $r_1,\dots,r_n$ be given positive integers and $e_0,\dots,e_n$ be real numbers with

$$(2)\qquad e_0\dots e_n\neq 0,\qquad \operatorname{sign} e_k=(-1)^{s_k},\qquad k=0,\dots,n,$$

where $s_k=r_{k+1}+\dots+r_n$, $k=0,\dots,n-1$, $s_n:=0$, $s:=s_0+\dots+s_n$.

Then there exist an unique system of points $\bar{x}=(x_1,\dots,x_n)\ \Omega_n$ and a constant $c\neq 0$, such that the Blashke product $B(\bar{x};t)$ of the form (1) satisfies

$$(3)\qquad \int_{x_k}^{x_{k+1}} B(\bar{x};t)dt=ce_k,\qquad k=0,\dots,n,$$

$x_0:=-1$, $x_{n+1}:=1$.

Proof. Form the system

(4) $$f_k(c,x_1,\ldots,x_n) = \int_{x_k}^{x_{k+1}} B(\bar{x};t)dt - ce_k = 0, \qquad k=0,\ldots,n$$

of non-linear equations with respect to $(c,\bar{x})$ and denote by J the Jacobian of (4)

$$J = D(f_0,\ldots,f_n) \,/\, D(c,x_1,\ldots,x_n).$$

First we show that $J \neq 0$ at the points $(c,x_1,\ldots,x_n,e_0,\ldots,e_n)$ satisfying (2) and (4). Note that

$$\frac{\partial f_k(c,\bar{x})}{\partial c} = -e_k, \qquad k=0,\ldots,n,$$

$$\frac{\partial f_k(c,\bar{x})}{\partial x_j} = \int_{x_k}^{x_{k+1}} \frac{\partial B(\bar{x};t)}{\partial x_j}dt, \qquad k=0,\ldots,n, \qquad j=1,\ldots,n,$$

$$\frac{\partial B(\bar{x};t)}{\partial x_j} = r_j B(\bar{x};t)(t^2-1)u_j(t), \qquad j=1,\ldots,n.$$

Then

$$J = \begin{vmatrix} -e_0 & \int_{x_0}^{x_1} r_1 B(\bar{x};t)(t^2-1)u_1(t)dt & \ldots & \int_{x_0}^{x_1} r_n B(\bar{x};t)(t^2-1)u_n(t)dt \\ \ldots & \ldots & \ldots & \ldots \\ -e_k & \int_{x_k}^{x_{k+1}} r_1 B(\bar{x};t)(t^2-1)u_1(t)dt & \ldots & \int_{x_k}^{x_{k+1}} r_n B(\bar{x};t)(t^2-1)u_n(t)dt \\ \ldots & \ldots & \ldots & \ldots \\ -e_n & \int_{x_n}^{x_{n+1}} r_1 B(\bar{x};t)(t^2-1)u_1(t)dt & \ldots & \int_{x_n}^{x_{n+1}} r_n B(\bar{x};t)(t^2-1)u_n(t)dt \end{vmatrix}$$

and hence

(5) $$J = -r_1\ldots r_n \sum_{k=0}^{n} (-1)^k e_k D_k,$$

where

$$D_k = \begin{vmatrix} \int_{x_0}^{x_1} B(\bar{x};t)(t^2-1)u_1(t)dt & \dots & \int_{x_0}^{x_1} B(\bar{x};t)(t^2-1)u_n(t)dt \\ \dots\dots & & \dots\dots \\ \int_{x_{k-1}}^{x_k} B(\bar{x};t)(t^2-1)u_1(t)dt & \dots & \int_{x_{k-1}}^{x_k} B(\bar{x};t)(t^2-1)u_n(t)dt \\ \int_{x_{k+1}}^{x_{k+2}} B(\bar{x};t)(t^2-1)u_1(t)dt & \dots & \int_{x_{k+1}}^{x_{k+2}} B(\bar{x};t)(t^2-1)u_n(t)dt \\ \dots\dots & & \dots\dots \\ \int_{x_n}^{x_{n+1}} B(\bar{x};t)(t^2-1)u_1(t)dt & \dots & \int_{x_n}^{x_{n+1}} B(\bar{x};t)(t^2-1)u_n(t)dt \end{vmatrix}.$$

Fix an integer k, $0\le k\le n$ and assume that $D_k=0$ for some $\bar{x}\in\Omega_n$. Then from the linear dependence of the columns of D_k it follows that there exists a function $u \in \operatorname{span}\{u_1,\dots,u_n\}\setminus\{0\}$ which satisfies the equalities

$$\int_{x_j}^{x_{j+1}} B(\bar{x};t)(t^2-1)u(t)dt = 0 , \qquad j=0,\dots,k-1,k+1,\dots,n.$$

But $B(\bar{x};t)(t^2-1) \neq 0$ for all $t\in A$. Therefore u must change its sign at least once in each interval (x_j,x_{j+1}), $j=0,\dots,k-1,k+1,\dots,n$. That is, u has at least n zeros in A, which contradicts the Lemma. So, $D_k\neq 0$ for all $\bar{x}\in\Omega_n$.

Similarly we get

$$D_k(y_1,\dots,y_k,z_{k+1},\dots,z_n) \neq 0 ,$$

$$D_k(y_1,\dots,y_k,z_{k+1},\dots,z_n) := \det \Big(\int_{y_i}^{z_i} B(\bar{x};t)(t^2-1)u_j(t)dt \Big)_{i,j=1}^{n} ,$$

for arbitrary $y_1,\dots,y_k,z_{k+1},\dots,z_n$ with

$$-1<y_1<z_1=x_1<y_2<z_2=x_2<\dots<y_k<z_k=x_k<x_{k+1}=y_{k+1}<z_{k+1}<\dots<x_n=y_n<z_n<1.$$

Consider

$$D_k(h) := D_k(y_1,\dots,y_k,z_{k+1},\dots,z_n)$$

for $\qquad y_i=x_i-h, \qquad i=1,\dots,k \qquad$ and $\qquad z_i=x_i+h, \qquad i=k+1,\dots,n.$

We now show that $D_k(h)$ becomes with dominant main diagonal for sufficiently small positive h, and we even find $\operatorname{sign} D_k$. Let us fix an integer j, $1\le j\le n$

and set $h_0 = \frac{1}{2} \min\{ |x_{i+1} - x_i| : i=0,\dots,n \}$. Then

$$\sum_{i=1, i\neq j}^{n} |u_i(t)| \,/\, |u_j(t)| \le dh \qquad \text{for all} \quad t\in(y_j, z_j),$$

$$d := (n-1) \,/\, [2h_0^2(2-h_0)] = \text{const}.$$

Hence

$$\sum_{i=1, i\neq j}^{n} |B(\bar{x};t)(t^2-1)u_i(t)| \le dh|B(\bar{x};t)(t^2-1)u_j(t)| , \qquad t\in(y_j, z_j).$$

It is not difficult to see that

$$\sum_{i=1, i\neq j}^{n} \left| \int_{y_j}^{z_j} B(\bar{x};t)(t^2-1)u_i(t)dt \right| \Big/ \left| \int_{y_j}^{z_j} B(\bar{x};t)(t^2-1)u_j(t)dt \right| \le dh$$

and for sufficiently small h, $0<h<h_0$, $D_k(h)$ has dominant main diagonal. Then

$$\operatorname{sign} D_k = \operatorname{sign} D_k(h) = \prod_{j=1}^{n} \operatorname{sign} \left[\int_{y_j}^{z_j} B(x;t)(t^2-1)u_j(t)dt \right]$$

$$= \prod_{j=0}^{k-1} (-1)^{s_j} \cdot \prod_{j=k+1}^{n} (-1)^{1+s_j} = (-1)^{n-k+s-s_k} ,$$

and

(6) $$\operatorname{sign} (-1)^k e_k D_k = (-1)^{n+s} , \qquad k=0,\dots,n.$$

The equalities (5) and (6) imply $J\neq 0$ for arbitrary $(c,\bar{x},e_0,\dots,e_n)$ satisfying (2), (4).

Let $\bar{x}^0 = (x_1^0,\dots,x_n^0) \in \Omega_n$, $x_0^0 := -1$, $x_{n+1}^0 := 1$,

$$e_k^0 = \int_{x_k^0}^{x_{k+1}^0} B(\bar{x}^0;t)dt , \qquad k=0,\dots,n.$$

Consider the system of differential equations

(7) $$\frac{d}{dv} \int_{x_k(v)}^{x_{k+1}(v)} B(\bar{x}(v);t)dt = e_k \frac{dc(v)}{dv} - e_k^0 , \qquad k=0,\dots,n$$

with initial conditions $c(0)=0$, $\bar{x}(0)=\bar{x}^0$.

Integrating (7) we find

$$(8)\qquad \int_{x_k(v)}^{x_{k+1}(v)} B(\bar{x}(v);t)dt = e_k c(v) + (1-v)e_k^0$$

and at $v=1$ we get (3). Our goal now is to show that the solution to (7) can be extended to $[0,1]$ keeping the properties :

(i) $\bar{x}(v) \in \Omega_n$;

(ii) the right hand side of (8) does not vanish for $v \in [0,1]$.

We have

$$(9)\qquad -e_k \frac{dc(v)}{dv} + \sum_{j=1}^{n} \left[\int_{x_k(v)}^{x_{k+1}(v)} \frac{\partial B(\bar{x}(v);t)}{\partial x_j} dt \right] \frac{dx_j(v)}{dv} = -e_k^0, \qquad k=0,\dots,n.$$

Considering (9) as a system of linear equations with respect to

$$\frac{dc(v)}{dv}, \quad \frac{dx_j(v)}{dv}, \qquad j=1,\dots,n$$

we can find $\frac{dc(v)}{dv}$ by Kramer's formula, since the determinant of (9) is just Jacobian (5), which is non-zero for $x \in \Omega_n$. Thus

$$0 < \frac{\min_k |e_k^0|}{\max_k |e_k|} \le \frac{dc(v)}{dv} = \frac{\sum_k (-1)^k e_k^0 D_k \big|_{\bar{x}=\bar{x}(v)}}{\sum_k (-1)^k e_k D_k \big|_{\bar{x}=\bar{x}(v)}} = \frac{\sum_k |e_k^0| \cdot |D_k|_{\bar{x}=\bar{x}(v)}|}{\sum_k |e_k| \cdot |D_k|_{\bar{x}=\bar{x}(v)}|}$$

$$\le \frac{\max_k |e_k^0|}{\min_k |e_k|}.$$

Therefore $c(v)$ is bounded, positive and monotone increasing for $v \in (0,1]$ and because of

$$\operatorname{sign} e_k = \operatorname{sign} e_k^0, \qquad k=0,\dots,n,$$

(ii) holds. Now it easy follows from (8) and (ii) that there is no sequence of solutions $(c(v_m),\bar{x}(v_m))$, $v_m \in [0,1]$ with

$$\lim_{m\to\infty} \bar{x}(v_m) \in \partial\Omega_n.$$

This completes the proof of the uniqueness part of the theorem.

Consider the mapping

$$F : \Omega_n \to W, \qquad W = \{ (c,\bar{x}) : (c,\bar{x}) \text{ solves (3)}, c>0, \bar{x}\in\Omega_n \},$$

which transforms each $\bar{x}^0 \in \Omega_n$ into the solution to (7) at $v=1$. Observe that

(a) F is continious, since by a classical result from the theory of differential equations the solution $(c(v),\bar{x}(v))$ to (7) depends continiously on the initial conditions $c(0)=0$, $\bar{x}(0)=\bar{x}^0$. Ω_n is connected, therefore $F(\Omega_n)$ is connected too. Further, if $(c,\bar{x})$ is a solution to (3), then it is easy to check that $c(v)=vc$, $\bar{x}(v)\equiv\bar{x}$ is a solution to the system of differential equations with initial conditions $c(0)=0$, $\bar{x}(0)=\bar{x}$. Moreover, by the uniqueness it is the only solution. Thus $F(\Omega_n)=W$ and W is connected.

(b) W consists of isolated points, which follows from the Implicit Function Theorem.

By (a) W is connected and (b) implies that it consists of one point only. The theorem is proved.

The differential equations approach used in the proof was applied firstly by C. Fitzgerald, L. Schumaker [4] and later by R. Barrar, H. Loeb [1], B. Bojanov and G. Grozev [2]. Similar problems was solved by B. Bojanov [3] for polynomials and by R. Barrar, H. Loeb [1] for generalized polynomials.

We conclude with the remark that the Theorem holds if in (3) we have

$$\int_{x_k}^{x_{k+1}} w(t)|B(\bar{x};t)|^p\, dt = ce_k, \qquad k=0,\dots,n,$$

where $1\le p<\infty$, $w(t)$ is a strict positive continious in $[-1,1]$ function and

$$e_k>0, \qquad k=0,\dots,n.$$

References

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