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SOME INVERSE THEOREMS FOR A CERTAIN SPECIAL CLASS OF OPERATORS

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1. Introduction. The purpose of this note is to report some inverse theorems for various positive linear operators of the form

$$L_n(f,x)=\sum_{k=0}^{N}\lambda_{n,k+1}(x)\int_0^B\omega_{n,k}(t)f(t)\,dt, \tag{1}$$

where N and B may be finite or infinite. Detailed proofs of our results are too lengthy to be included, and a related paper will appear in "Demonstratio Mathematica" in 1988. This note also gives a characterization of the class $Lip^*\alpha$ of periodic functions using Rappoport's discrete Vallée-Poussin operator.

As usual denote

$$p_{nk}(x)=\binom{n}{k}x^k(1-x)^{n-k},$$
$$b_{nk}(x)=\binom{n+k-1}{k}x^k(1+x)^{-(n+k)},$$
$$S_{nk}(x)=e^{-nx}(nx)^k/k!\ .$$

We are concerned with the linear smoothing operators

$$A_n(f,x)=\sum_{k=0}^{n-2}p_{n,k+1}(x)(n-1)\int_0^1 p_{n-2,k}(t)f(t)\,dt, \tag{2}$$

$$B_n(f,x)=\sum_{k=0}^{\infty}b_{n,k+1}(x)(n+1)\int_0^\infty b_{n+2,k}(t)f(t)\,dt, \tag{3}$$

$$C_n(f,x)=\sum_{k=0}^{\infty}S_{n,k+1}(x)\,n\int_0^\infty S_{n,k}(t)f(t)\,dt, \tag{4}$$

$$D_n(f,x)=\sum_{k=0}^{\infty}b_{n,k+1}(x)\,n\int_0^\infty S_{n,k}(t)f(t)\,dt. \tag{5}$$

These operators may be written in the unified form (1), in which N and B denote n-2 and 1 for (2), and ∞ for (3), (4) and (5). Clearly (4) is generated from Szasz-Mirakyan operator, and (5) is of compound type.

As may be observed, (2) and (3) are slight modifications of those operators investigated by Derriennic [2] and Sahal and Pasad [4] respectively. In fact, no inverse theorem could be proved for the original operators. In a like manner, for periodic functions of period 2π we introduce the following three operators

$$(6) \qquad F_n(f, x) = \sum_{k=0}^{2n} V_{nk}(x) \frac{2n+1}{2\pi} \int_0^{2\pi} V_{nk}(t) f(t) dt,$$

$$(7) \qquad G_n(f, x) = \sum_{k=0}^{4n-1} J_{nk}(x) \frac{2n}{\pi} \int_0^{2\pi} J_{nk}(t) f(t) dt,$$

$$(8) \qquad H_n(f, x) = \sum_{k=0}^{2n-1} K_{nk}(x) \frac{n}{\pi} \int_0^{2\pi} K_{nk}(t) f(t) dt,$$

where V, J and K are defined by the following (with $x \in [0, 2\pi]$)

$$V_{nk}(x) = \frac{2^{2n}(n!)^2}{(2n+1)!} \cos^{2n} \frac{1}{2}\left(x - \frac{2k\pi}{2n+1}\right),$$

$$J_{nk}(x) = \frac{3}{4n^2(2n^2+1)} \left(\frac{\sin \frac{n}{2}\left(x - \frac{k\pi}{2n}\right)}{\sin \frac{1}{2}\left(x - \frac{k\pi}{2n}\right)} \right)^4,$$

$$K_{nk}(x) = \left(\frac{1}{n} \sin \frac{\pi}{n}\right)^2 \left(\frac{\cos \frac{n}{2}\left(x - \frac{k\pi}{n}\right)}{\cos\left(x - \frac{k\pi}{n}\right) - \cos \frac{\pi}{n}} \right)^2,$$

which are known as the discrete Vallée-Poussin kernel and Bojanic-Shisha's discrete Jackson kernel, respectively.

Finally, we are also interested in obtaining an inverse theorem for Rappoport's operator

$$(9) \qquad R_n(f, x) = \frac{(2n)!!}{(2n+1)!!} \sum_{k=0}^{2n} f(x_k) \cos^{2n}\left(\frac{x_k - x}{2}\right),$$

where $x_k = 2k\pi/(2n+1)$, $k = 0, 1, \cdots, 2n$, and m!! denotes the double factorial, e.g. 5!! =1·3·5, 6!! =2·4·6.

2. Statement of theorems. Denote by

$$\Delta_h^2 f(x) = f(x+h) - 2f(x) + f(x-h), \qquad x \in [h, B-h].$$

In case B = ∞ , $[0, B]$ will be replaced by $[0, \infty)$. Also, we denote

$$\omega_2(f,\delta) = \sup_{0<h\leqslant\delta} \sup_x \left| \Delta_h^2 f(x) \right|,$$

$$Lip^*\alpha := \left\{ f \in C[0,B],\ \omega_2(f,\delta) = O(\delta^\alpha),\ \delta \to 0+ \right\}.$$

and we denote by $\|f\|$ the Chebyshev norm of f on the interval considered. In what follows M is always used to denote a positive constant, independent of x and n (n =1,2,3,...).

Theorem 1. Let L_n represent any of the operators A_n, B_n, C_n, and D_n . For $f \in C[0,B]$, $\alpha \in (0,2)$, the estimation

$$\left| L_n(f,x) - f(x) \right| \leqslant M \cdot \left(\frac{\varphi(x)}{n} \right)^{\alpha/2} \qquad (\forall n)$$

holds if and only if $f \in Lip^*\alpha$, where $\varphi(x)=x(1-x)$ for A_n, $\varphi(x)=x(1+x)$ for B_n, $\varphi(x)=x$ for C_n, and $\varphi(x)=\frac{1}{2}x(2+x)$ for D_n.

Theorem 2. Let $f \in C_{2\pi}$, $\alpha \in (0,2)$. Then the following estimation

$$\left| F_n(f,x) - f(x) \right| \leqslant M \cdot n^{-\alpha/2} \qquad (\forall n)$$

is equivalent to the assertion $f \in Lip^*\alpha$ with $[0,B] \equiv [0,2\pi]$.

Theorem 3. Let $f \in C_{2\pi}$, $\alpha \in (0,2)$. Then the relation

$$\left| G_n(f,x) - f(x) \right| \leqslant M \cdot n^{-\alpha}$$

is equivalent to the condition $f \in Lip^*\alpha$ with $[0,B] \equiv [0,2\pi]$.

There is a complete analogue of Theorem 3 for the operator H_n . Moreover, for periodic functions $f \in C_{2\pi}$ we define in a like manner

$$Lip^*\alpha := \left\{ f \in C_{2\pi},\ \omega_2(f,\delta) = O(\delta^\alpha),\ \delta \to 0+ \right\}$$

in which $\omega_2(f,\delta)$ has the same meaning as before. Then we have

Theorem 4. For $f \in C_{2\pi}$ the estimation

$$\left| R_n(f,x) - f(x) \right| \leqslant M \cdot n^{-\alpha/2}$$

holds if and only if $f \in Lip^*\alpha$, ($0<\alpha<2$).

3. Sketch on the method of proof. For the operators L_n considered in Theorem 1 one may verify at once the following

$$\frac{1}{2} L_n((t-x)^2, x) = \frac{\varphi(x)}{n} + o\left(\frac{\varphi(x)}{n}\right),$$

$$L_n((t-x), x) = o\left(\frac{\varphi(x)}{x}\right), \quad L_n(1, x) = 1 + o\left(\frac{\varphi(x)}{x}\right).$$

Thus the "if" part of Theorem 1 can be verified along the similar lines as in [5]. The "only if" part can be proved by using the idea due to Berens and Lorentz [1] and also the following two lemmas

Lemma 1. If $f \in C^2[0, B]$, $x \in (0, B)$, then

$$\left|\left(L_n(f, x)\right)''\right| \leqslant M \cdot \|f''\|.$$

Lemma 2. For $f \in C[0, B]$, $x \in (0, B)$, we have

$$\left|\left(L_n(f, x)\right)''\right| \leqslant M \cdot \|f\| \frac{n}{\psi(x)},$$

where $\psi(x) = \varphi(x)$ for A_n, B_n, C_n; and $\psi(x) = x(1+x)$ for D_n.

in addition to Satô's Lemma 2 of his paper [5].

Theorems 2 and 3 can be proved in a somewhat similar manner. Also, the proof of the "only if" part of Theorem 4 has to make use of the lemma, $\left|\left(R_n(f, x)\right)''\right| \leqslant 3\|f''\|$, and the useful idea of [1].

References

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