CONSTRUCTIVE THEORY OF FUNCTIONS'87

SOFIA, 1988

A NOTE ON N-DIMENSIONAL HARDY'S INEQUALITY

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We shall deal with a special type of the inequality

$$\left(\int_\Omega |u(x)|^q\, w(x)\, dx\right)^{1/q} \le c\left(\sum_{i=1}^{N} \int_\Omega \left|\frac{\partial u}{\partial x_i}(x)\right|^p v_i(x)\, dx\right)^{1/p}, \tag{1}$$

where $1 \le p$, $q < \infty$, Ω is a domain in $\mathbb{R}^N$, the functions $w, v_1, \dots, v_N$ are *weights*, i.e. measurable, a.e. on Ω positive and finite functions.

The problem is to establish sufficient and necessary conditions on the weights $w, v_1, \dots, v_N$ under which inequality (1) is valid for a rather wide class of functions $u = u(x)$ with a positive constant $c > 0$ independent of u.

The problem was solved for $N = 1$ (see [1], [3], [8] and [10]).

For general $N \ge 1$ there exist only partial results concerning our problem. One of those results is that presented at the previous conference by B. OPIC and A. KUFNER [13]. They found sufficient conditions for weights $w, v_1, \dots, v_N$ which guarantee the validity of inequality (1) in the case $1 < p = q < \infty$.

Other approaches can be found in [2], [4], [6] and [7] (only sufficient conditions) or in [10] (conditions are sufficient and necessary but they are very complicated to verify).

Here we shall show another approach. Using results from [9] and from [5], [11] we can immediately derive simple sufficient and necessary conditions under which inequality (1) for power-type weights holds.

Let us first recall the results that we shall use.

<u>LEMMA 1.</u> *Let us suppose* $\Omega \in C^{0,1}$ *(i.e.* Ω *is a bounded domain with a Lipschitzian boundary* $\partial\Omega$ *),* $1 \le p < \infty$, $\varepsilon < p - 1$. *Then the norms* $\|\cdot\|_{1,p,\Omega,d^\varepsilon,d^\varepsilon}$ *and* $|||\cdot|||_{1,p,\Omega,d^\varepsilon}$, *where*

(2) $$\|u\|_{1,p,\Omega,d^\varepsilon,d^\varepsilon} = \left(\int_\Omega |u(x)|^p\, d^\varepsilon(x)\, dx + \int_\Omega |\nabla u(x)|^p\, d^\varepsilon(x)\, dx\right)^{1/p},$$

(3) $$|||u|||_{1,p,\Omega,d^\varepsilon} = \left(\int_\Omega |\nabla u(x)|^p\, d^\varepsilon(x)\, dx\right)^{1/p}$$

($|\nabla u(x)|^p = \sum_{i=1}^N \left|\frac{\partial u}{\partial x_i}(x)\right|^p$, $d(x) = \mathrm{dist}\,(x,\partial\Omega)$) *are equivalent on the space* $C_0^\infty(\Omega)$.

For the p r o o f see [9], Proposition 9.2.

<u>LEMMA 2.</u> *Let us suppose that* $\Omega \in C^{0,1}$.

(i) *If* $1 \le p \le q < \infty$, $1/N \ge 1/p - 1/q$, $\beta \ne p - 1$, *then*

(4) $$W_0^{1,p}(\Omega; d^\beta, d^\beta) \subsetneq L^q(\Omega; d^\alpha)$$

if and only if

(5) $$N\left(\frac{1}{q} - \frac{1}{p}\right) + \frac{\alpha}{q} - \frac{\beta}{p} + 1 \ge 0 .$$

(ii) *If* $1 \le q < p < \infty$ *then imbedding* (4) *holds if and only if*

(6) $$\left(\frac{1}{q} - \frac{1}{p}\right) + \frac{\alpha}{q} - \frac{\beta}{p} + 1 > 0 .$$

For the p r o o f see [5], [11] (cf. [12]).

Combining these two lemmas we get.

<u>THEOREM</u> (N-dimensional Hardy's inequality). *Let* $\Omega \in C^{0,1}$, $1 \le p, q < \infty$, $\beta < p - 1$. *Then there exists a positive constant* $c > 0$ *such that the inequality*

(7) $$\left(\int_\Omega |u(x)|^q\, d^\alpha(x)\, dx\right)^{1/q} \le c \left(\int_\Omega |\nabla u(x)|^p\, d^\beta(x)\, dx\right)^{1/p}$$

holds for all $u \in C_0^\infty(\Omega)$ *if and only if*

(i) $1 \le p \le q < \infty$, $1/N \ge 1/p - 1/q$, $N\left(\frac{1}{q} - \frac{1}{p}\right) + \frac{\alpha}{q} - \frac{\beta}{p} + 1 \ge 0$

or

(ii) $1 \le q < p < \infty$, $\left(\frac{1}{q} - \frac{1}{p}\right) + \frac{\alpha}{q} - \frac{\beta}{p} + 1 > 0$.

<u>REMARK.</u> The Theorem may be generalized for $\Omega \in C^{0,\kappa}$, $0 < \kappa \le 1$ (see [11]). Unfortunately, for $0 < \kappa < 1$, necessary conditions are different from sufficient ones.

References

[1] J. S. BRADLEY : *Hardy inequality with mixed norms*. Canad. Math. Bull. 21 (1978), 405-408

[2] E. FABES & C. KENIG & R. SERAPIONI : *The local regularity of solutions of degenerate elliptic equations*. Comm. Partial Differential Equations, 7 (1982), 77-116, MR 84i:35070

[3] P. GURKA : *Generalized Hardy's inequality*. Časopis Pěst. Mat., 109 (1984), 194-203, MR 85m:26019

[4] P. GURKA & A. KUFNER : *A note on a two-weighted Sobolev inequality*. Banach Center Publications, 27th semester, Approx. Theory and Function Spaces, February - May 1986

[5] P. GURKA & B. OPIC : *Continuous and compact imbeddings of weighted Sobolev spaces I*. (To appear in Czechoslovak Math. J.)

[6] E. HARBOURE : *Two weighted Sobolev and Poincaré inequalities and some applications* (preprint)

[7] S. CHANILLO & R. L. WHEEDEN : *Weighted Poincaré and Sobolev inequalities and estimates for weighted Peano maximal functions*. Amer. J. Math., 107 (1985), 1191-1226

[8] V. M. KOKILAŠVILI : *On Hardy's inequalities in weighted spaces*. (Russian), Bull. Acad. Sci. Georgian SSR, 96 (1979), №. 1

[9] A. KUFNER : *Weighted Sobolev spaces*. John Wiley & Sons, Chichester - New York - Brisbane - Toronto - Singapore 1985

[10] V. G. MAZ'JA : *Sobolev spaces*. (Russian). Izdatelstvo Leningradskogo universiteta, Leningrad 1985

[11] B. OPIC & P. GURKA : *Continuous and compact imbeddings of weighted Sobolev spaces II*. (To appear in Czechoslovak Math. J.)

[12] B. OPIC & P. GURKA : *On imbeddings of weighted Sobolev spaces*. Constructive function theory '87. Proceedings of the Internal. Conf. Varna, May 25 - 31, 1987. Publ. House of the Bulg. Acad. Sci., Sofia, 1987

[13] B. OPIC & A. KUFNER : *Some inequalities in weighted Sobolev spaces*. Constructive function theory '84. Proceedings of the Internal. Conf. Varna, May 27 - June 2, 1984. Publ. House of the Bulg. Acad. Sci., Sofia 1984, 644-648.

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