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COMMUTATIVITY OF OPERATORS FROM BASKAKOV-DURRMEYER TYPE

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1. Introduction and definition of the operators

For a function $f \in L_1[0,1]$ the n-th, $n \in \mathbb{N}$, Bernstein-Durrmeyer operator B_n is given by

$$(1.1)\quad (B_n f)(x) = (n+1)\sum_{k=0}^{n} p_{nk}(x)\int_0^1 p_{nk}(t)f(t)dt\ ,\ x \in [0,1],$$

where

$$p_{nk}(x) = \binom{n}{k}x^k(1-x)^{n-k}\ .$$

These operators B_n were first defined and considered by Durrmeyer [6] and studied by Derriennic [4]. The saturation class was determined by the author [8]. Direct and inverse results for these operators, their derivatives and linear combinations were discussed by Ditzian and Ivanov [5]. With the aid of the representation of $B_n f$ in terms of the eigenfunctions of the operator (see [4]) they stated the commutativity, i.e.

$$B_n(B_m f) = B_m(B_n f)$$

for all $f \in L_1[0,1]$, $n,m \in \mathbb{N}$, which was very useful for the proof of an inverse result for the derivatives of $B_n f$ (see [5], Theorem 7.1). Since neither the classical Bernstein-, Szász-Mirakjan- and Baskakov operators nor their Kantorovič variants possesses this nice property it is interesting to consider the Durrmeyer construction from a more general point of view. Following Baskakov [2] we consider a sequence $(\phi_n)_{n\in\mathbb{N}}$ of real valued functions with the following properties on an interval $[0,b]$, $b > 0$.

(1.2) $\phi_n \in C^\infty$

(1.3) $\phi_n(0) = 1$

(1.4) $(-1)^k \phi_n^{(k)} \geq 0$ for $k \in \mathbb{N}_0$

(1.5) $\exists\, c \in \mathbb{Z} : \phi_n^{(k+1)} = -n\phi_{n+c}^{(k)}$ for $k \in \mathbb{N}_0$ and $n+c > 0$

In [11], Chapter 4.1 suitable sequences of functions $(\phi_n)_{n\in\mathbb{N}}$ are stated. For $c = 0$, $\phi_n(x) = e^{-nx}$ and $c > 0$, $\phi_n(x) = (1+cx)^{-n/c}$ the conditions (1.2) - (1.5) hold for all $x \in [0,\infty)$. From this we obtain the following definition of operators from Baskakov-Durrmeyer Type for $c \geq 0$.

Definition 1.1

For $f \in L_p[0,\infty)$, $1 \leq p \leq \infty$, $M_n f$, $n \in \mathbb{N}$, $n > c$ is defined by

$$(M_n f)(x) = (n-c) \sum_{k=0}^{\infty} p_{nk}(x) \int_0^\infty p_{nk}(t) f(t)\,dt, \quad x \in [0,\infty)$$

where

$$p_{nk}(x) = \frac{(-1)^k}{k!} \cdot x^k \phi_n^{(k)}(x) ,$$

$$\phi_n(x) = \begin{cases} e^{-nx} & \text{for } c = 0 \\ (1+cx)^{-n/c} & \text{for } c > 0 \end{cases} .$$

We notice that the analogous definition for $c = -1$, $\phi_n(x) = (1-x)^n$ for the interval $[0,1]$ leads to the Bernstein-Durrmeyer operators (1.1). For $c = 0$ the operator was defined by Mazhar and Totik in [10].

2. The commutativity result

We are now able to state the commutativity result for the operators of Baskakov-Durrmeyer type.

Theorem 2.1

For every $f \in L_p[0,\infty)$, $1 \leq p \leq \infty$

$$M_n(M_m f) = M_m(M_n f)$$

holds for arbitrary $n,m \in \mathbb{N}$, $n,m > c$.

Proof

We first sketch out the main steps of our proof.

(2.1) 1.step: For $x \in [0,\infty)$ we show the integral representations

$$M_n(M_m f)(x) = \frac{(n-c)(m-c)}{(n+m-c)} \int_0^\infty f(y) G_{nm}(x,y)\,dy$$

and

$$M_m(M_n f)(x) = \frac{(n-c)(m-c)}{(n+m-c)} \int_0^\infty f(y) G_{mn}(x,y)\,dy$$

respectively with

$G_{nm}(x,y)$

$$= \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} p_{nk}(x) p_{mj}(y) \binom{k+j}{j} \prod_{l=0}^{k-1}(n+cl) \prod_{l=0}^{j-1}(m+cl) \prod_{l=0}^{k+j-1}(n+m+cl)^{-1}$$

and

$G_{mn}(x,y)$

$$= \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} p_{nk}(y) p_{mj}(x) \binom{k+j}{j} \prod_{l=0}^{k-1}(n+cl) \prod_{l=0}^{j-1}(m+cl) \prod_{l=0}^{k+j-1}(n+m+cl)^{-1}$$

respectively.

(2.2) 2.step: We consider G_{nm} and G_{mn} as functions of two complex variables and show that they are holomorphic in a certain region.

(2.3) 3.step: We show the equality of G_{nm} and G_{mn} in an open neighbourhood of (0,0) by considering the Taylor series at (0,0).

(2.4) 4.step: We finish our proof by using the identity theorem for analytic functions from which we conclude the equality $G_{nm}(x,y) = G_{mn}(x,y)$ for all $x,y \in [0,\infty)$.

Proof of (2.1)

We have

$$M_n(M_m f)(x) = (n-c)\sum_{k=0}^{\infty} p_{nk}(x) \int_0^\infty p_{nk}(t) \left\{(m-c)\sum_{j=0}^{\infty} p_{mj}(t) \int_0^\infty p_{mj}(y) f(y)\,dy\right\} dt$$

$$= \int_0^\infty f(y) \left\{ \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} p_{nk}(x) p_{mj}(y) (n-c)(m-c) \int_0^\infty p_{nk}(t) p_{mj}(t)\,dt \right\} dy$$

where the interchanging of integration and summation is allowed by a corollary of Lebesgue's dominated convergence theorem (see [9], Corollary 12.33).

By the definition of p_{nk} and p_{mj} we have

$$p_{nk}(t)p_{mj}(t) = \binom{k+j}{j}\prod_{l=0}^{k-1}(n+cl)\prod_{l=0}^{j-1}(m+cl)\prod_{l=0}^{k+j-1}(n+m+cl)^{-1}p_{n+m,k+j}(t)$$

and

$$\int_0^{\infty} p_{n+m,k+j}(t)dt = (n+m-c)^{-1} .$$

Hence the integral representation for $M_n(M_m f)$ in (2.1) is valid and the analogous term for $M_n(M_m f)$ follows in the same manner.

<u>Proof of (2.2)</u>

Consider now G_{nm} and G_{mn} as functions of two complex variables x and y. Let $G = \left\{z\in\mathbb{C}:\ |z|<\frac{1}{2c} \text{ or } \left[Re(z)\in[0,\infty) \text{ and } Im(z)\in\left(-\frac{1}{2c},\frac{1}{2c}\right)\right]\right\}$ for $c > 0$ and $G = \mathbb{C}$ for $c = 0$ and $(x,y) \in G\times G$.

In the <u>case c = 0</u> we have

$$\binom{k+j}{j}\frac{n^k m^j}{(n+m)^{k+j}} \le 1$$

by using the binomial formula for $(n+m)^{k+j}$. From this we get

$$\left|\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}p_{nk}(x)p_{mj}(y)\binom{k+j}{j}\frac{n^k m^j}{(n+m)^{k+j}}\right|$$

$$\le \sum_{k=0}^{\infty}|p_{nk}(x)|\sum_{j=0}^{\infty}|p_{mj}(y)|$$

$$= \sum_{k=0}^{\infty}\frac{(n|x|)^k}{k!}|e^{-nx}|\sum_{j=0}^{\infty}\frac{(m|y|)^j}{j!}|e^{-my}|$$

$$= e^{-nRe(x)}\cdot e^{n|x|}\cdot e^{-mRe(y)}\cdot e^{m|y|}$$

For the <u>case c > 0</u> let $N = n/c$ and $M = m/c$. By the definition of the Γ-function and by use of Stirling's formula and the binomial formula we get

$$\binom{k+j}{j}\prod_{l=0}^{k-1}(n+cl)\prod_{l=0}^{j-1}(m+cl)\prod_{l=0}^{k+j-1}(n+m+cl)^{-1}$$

$$= \binom{k+j}{j}\cdot\frac{\Gamma(N+k)\Gamma(M+j)}{\Gamma(N+M+k+j)}\cdot\frac{\Gamma(N+M)}{\Gamma(N)\Gamma(M)}$$

$$\leq \binom{k+j}{j}\cdot e^{3}\cdot\frac{(N+k)^{k}(M+j)^{j}}{(N+M+k+j)^{k+j}}\cdot\frac{(N+M)^{N+M}}{N^{N}M^{M}}$$

$$\leq e^{3}\cdot\frac{(N+M)^{N+M}}{N^{N}M^{M}}\quad .$$

Hence

$$\left|\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}p_{nk}(x)p_{mj}(y)\binom{k+j}{j}\prod_{l=0}^{k-1}(n+cl)\prod_{l=0}^{j-1}(m+cl)\prod_{l=0}^{k+j-1}(n+m+cl)^{-1}\right|$$

$$\leq e^{3}\cdot\frac{(N+M)^{N+M}}{N^{N}M^{M}}\sum_{k=0}^{\infty}|p_{nk}(x)|\sum_{j=0}^{\infty}|p_{mj}(y)|$$

$$= e^{3}\cdot\frac{(N+M)^{N+M}}{N^{N}M^{M}}\sum_{k=0}^{\infty}\frac{|x|^{k}}{k!}|1+cx|^{-\frac{n+kc}{c}}\prod_{l=0}^{k-1}(n+cl)$$

$$\sum_{j=0}^{\infty}\frac{|y|^{j}}{j!}|1+cy|^{-\frac{m+jc}{c}}\prod_{l=0}^{j-1}(m+cl)$$

$$= e^{3}\cdot\frac{(N+M)^{N+M}}{N^{N}M^{M}}\sum_{k=0}^{\infty}\frac{(-c|x|)^{k}}{k!}|1+cx|^{-N-k}\prod_{l=0}^{k-1}(-N-l)$$

$$\sum_{j=0}^{\infty}\frac{(-c|y|)^{j}}{j!}|1+cy|^{-M-j}\prod_{l=0}^{j-1}(-M-l)$$

$$= e^{3}\cdot\frac{(N+M)^{N+M}}{N^{N}M^{M}}\Big[|1+cx|-|cx|\Big]^{-N}\cdot\Big[|1+cy|-|cy|\Big]^{-M}$$

by using the generalized binomial formula.

Hence $G_{n,m}(x,y)$ is absolutely uniformly convergent in a closed neighbourhood of every point $(x,y)\in G\times G$ for every $c\geq 0$. As the functions p_{nk} and p_{mj} are holomorphic in G we derive from the theorem of convergence by Weierstraß and Hartogs' theorem (see [3]) that G_{nm} is holomorphic in $G\times G$.

The same arguments hold for the holomorphy of G_{mn} in G×G. That means that G_{nm} and G_{mn} respectively can be expanded in a power series at every point $(x,y) \in G\times G$, being convergent for a certain open neighbourhood of (x,y) and equaling the functions there.

Proof of (2.3)

From now on we consider again G_{nm} and G_{mn} respectively as functions of real variables. From (2.2) we know that the Taylor series are convergent in a certain neighbourhood U(0,0) of the point (0,0) and equal the functions there. Thus for $(x,y) \in U(0,0)$ we have

$$G_{nm}(x,y) = \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{1}{(s+r)!} \binom{s+r}{s} x^r y^s \left[\frac{\partial^{r+s}}{\partial x^r \partial y^s} G_{nm}(x,y)\right](0,0) .$$

From this we see that for the equality of G_{nm} and G_{mn} in U(0,0) it is enough to show that $\left[\frac{\partial^{r+s}}{\partial x^r \partial y^s} G_{nm}(x,y)\right](0,0)$ is symmetric in n and m. Without loss of generality we consider the case $s \geq r$.

With Leibniz's formula we get

$$\left[\frac{\partial^r}{\partial x^r} p_{nk}(x)\right](0) = (-1)^k \cdot \sum_{\nu=0}^{r} \binom{r}{\nu} \left[\left(\frac{x^k}{k!}\right)^{(\nu)} (\phi_n^{(k)}(x))^{(r-\nu)}\right](0)$$

$$= \begin{cases} (-1)^k \phi_n^{(r)}(0) & \text{for } k \leq r \\ 0 & \text{for } k > r \end{cases}$$

$$= \begin{cases} (-1)^{k+r} \binom{r}{k} \prod_{l=0}^{r-1} (n+cl) & \text{for } k \leq r \\ 0 & \text{for } k > r \end{cases} .$$

Together with the definition of G_{nm} we get

$$\left[\frac{\partial^{r+s}}{\partial x^r \partial y^s} G_{nm}(x,y)\right](0,0)$$

$$= \prod_{l=0}^{r-1}(n+cl) \prod_{l=0}^{s-1}(m+cl) \sum_{k=0}^{r} \binom{r}{k} (-1)^k \cdot \frac{1}{k!}$$

$$\cdot \sum_{j=0}^{s} \binom{s}{j} (-1)^j \cdot \frac{(j+k)!}{j!} \prod_{l=0}^{k-1}(n+cl) \prod_{l=0}^{j-1}(m+cl) \prod_{l=0}^{k+j-1}(n+m+cl)^{-1} .$$

We will show that this equals

$$(2.5)\qquad \binom{r+s}{s}(-1)^{r+s}\prod_{l=0}^{r-1}(n+cl)(m+cl)\prod_{l=0}^{s-1}(n+cl)(m+cl)\prod_{l=0}^{r+s-1}(n+m+cl)^{-1},$$

which is symmetric in n and m, i.e. we prove the equality

$$(2.6)\qquad \sum_{k=0}^{r}\binom{r}{k}(-1)^{k}\cdot\frac{1}{k!}\sum_{j=0}^{s}\binom{s}{j}(-1)^{j}\cdot\frac{(j+k)!}{j!}$$

$$\cdot\prod_{l=0}^{k-1}(n+cl)\prod_{l=0}^{j-1}(m+cl)\prod_{l=0}^{k+j-1}(n+m+cl)^{-1}$$

$$=\binom{r+s}{s}\prod_{l=0}^{r-1}(m+cl)\prod_{l=0}^{s-1}(n+cl)\prod_{l=0}^{r+s-1}(n+m+cl)^{-1}.$$

We define

$$g_k(x)=\sum_{j=0}^{s}\binom{s}{j}(-1)^{j}\cdot\frac{(j+k)!}{j!}(1-x)^{j}\ .$$

With the binomial formula we have

$$x^{s}(x-1)^{k}=\sum_{j=0}^{s}\binom{s}{j}(x-1)^{j+k}$$

and

$$\frac{d^k}{dx^k}\Big[x^{s}(x-1)^{k}\Big]=\sum_{j=0}^{s}\binom{s}{j}(-1)^{j}\cdot\frac{(j+k)!}{j!}(1-x)^{j}=g_k(x)\ .$$

By Leibniz's formula for the k-th derivative of the product of two functions we get

$$(2.7)\qquad g_k(x)=\frac{d^k}{dx^k}\Big[x^{s}(x-1)^{k}\Big]=\sum_{j=0}^{k}\binom{k}{j}\binom{s}{j}k!x^{s-j}(-1)^{j}(1-x)^{j}\ .$$

To prove (2.6) we consider the two cases c = 0 and c > 0.

$\underline{c = 0}$

With $x = \frac{n}{m+n}$ and $(1-x) = \frac{m}{m+n}$ and the representation (2.7) for $g_k(x)$ (2.6) is reduced to the proposition

$$(2.8) \qquad \sum_{k=0}^{r} \binom{r}{k} (-1)^k \cdot \frac{1}{k!} x^k \sum_{j=0}^{k} \binom{k}{j} \binom{s}{j} k! x^{s-j} (-1)^j (1-x)^j$$

$$= \binom{r+s}{s} x^s (1-x)^r \quad .$$

Interchanging the order of summation, using the identity $\binom{r}{k}\binom{k}{j} = \binom{r}{j}\binom{r-j}{r-k}$ and indextransformation yield that the left hand side of (2.8) equals

$$x^s \sum_{j=0}^{r} \binom{s}{j}\binom{r}{j} (1-x)^j \sum_{k=0}^{r-j} \binom{r-j}{k} (-x)^{r-j-k}$$

$$= x^s \sum_{j=0}^{r} \binom{s}{j}\binom{r}{j} (1-x)^j (1-x)^{r-j}$$

$$= x^s (1-x)^r \sum_{j=0}^{r} \binom{s}{j}\binom{r}{j}$$

$$= \binom{r+s}{s} x^s (1-x)^r ,$$

where the last equations follow by the binomial formula and the identity $\sum_{j=0}^{r} \binom{s}{j}\binom{r}{j} = \binom{r+s}{s}$ (see [6], formula 3.1). Hence we have proved (2.8) and (2.6) respectively. That means (2.5) is true in the case c = 0.

$\underline{c > 0}$

With N = n/c and M = m/c, N,M > 1 as n,m > c, and the definition of the β-function (2.6) is reduced to the proposition

$$(2.9) \qquad \sum_{k=0}^{r} \binom{r}{k} (-1)^k \cdot \frac{1}{k!} \sum_{j=0}^{s} \binom{s}{j} (-1)^j \cdot \frac{(j+k)!}{j!} \beta(N+k, M+j)$$

$$= \binom{r+s}{s} \beta(M+r, N+s) \quad .$$

By using the integral representation

$$\beta(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1}dx$$

for the β-function (see [1]) we get that the left hand side of (2.9) equals

$$\int_0^1 \left\{ \sum_{k=0}^{r} \binom{r}{k}(-1)^k \cdot \frac{1}{k!} x^{N+k-1}(1-x)^{M-1} \sum_{j=0}^{s} \binom{s}{j}(-1)^j \cdot \frac{(j+k)!}{j!}(1-x)^j \right\} dx$$

$$= \int_0^1 \left\{ \sum_{k=0}^{r} \binom{r}{k}(-1)^k x^{N+k-1}(1-x)^{M-1} \sum_{j=0}^{k} \binom{k}{j}\binom{s}{j}(-1)^j x^{s-j}(1-x)^j \right\} dx$$

by our definition of the function g_k. With analogous transformations as in the case $c = 0$ we get that this equals

$$\int_0^1 \binom{r+s}{s} x^{N+s-1}(1-x)^{M+r-1}dx$$

$$= \binom{r+s}{s} \beta(N+s, M+r) \; .$$

Hence we have proved (2.9) and (2.6) respectively. That means (2.5) is also true for the case $c > 0$.

Proof of (2.4)

As $G_{nm}(x,y) = G_{mn}(x,y)$ in $U(0,0)$ by (2.3) and $G_{nm}(x,y)$ and $G_{mn}(x,y)$ are analytic for all $x,y \in (-\infty,\infty)$ in the case $c = 0$ and $x,y \in (\frac{-1}{2c},\infty)$ in the case $c > 0$ we can conclude $G_{nm}(x,y) = G_{mn}(x,y)$ for all $x,y \in [0,\infty)$ by use of the identity theorem for analytic functions (see [3]). Together with the integral representations (2.1) we get $M_n(M_m f)(x) = M_m(M_n f)(x)$ for all $x \in [0,\infty)$ what we wanted to prove.

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