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INTEGRAL REPRESENTATIONS OF FUNCTIONS IN WEIGHTED BERGMAN SPACES

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1. Introduction. Let w be a non-negative, locally integrable function defined on $\operatorname{Re} z>0$, then the weighted Hardy spaces H_w^p, $1 \le p < \infty$, with weight w, consist of those holomorphic functions in the half plane $\operatorname{Re} z > 0$, for which

$$\sup_{x>0} \int_{-\infty}^{\infty} |f(x+iy)|^p \, w(x+iy)dy < \infty$$

It was shown recently in [1] that if u and v belong to a certain weight class, and if $f \in H_v^p$, then there exists a function $F \in L_u^q$, $1 \le p \le q < \infty$, such that f is the Laplace transform of F, that is,

$$f(z) = (LF)(z) \doteq \int_0^{\infty} e^{-zt} F(t)dt , \qquad \operatorname{Re} z > 0 .$$

Moreover, as $x \to 0+$, f has boundary values.

The result together with its dual yields in the special case $p = 2$ and $u = v \equiv 1$ the familiar characterization of Paley and Wiener [5].

The purpose of this note is to prove analogous results for functions in $B_w^{p,q}$, $1 < p \le q < \infty$. Here $B_w^{p,q}$ denote the weighted Bergman spaces, with weight w, of functions f holomorphic in the half plane $\operatorname{Re} z > 0$, satisfying

$$\int_0^{\infty} \left[\int_{-\infty}^{\infty} |f(x+iy)|^p \, w(x+iy)dy \right]^{q/p} dx < \infty .$$

Our results depend on certain Fourier inequalities in weighted

L^p-spaces. The Fourier transform $\hat{f}$ of a function f is defined by

$$\hat{f}(x) = \int_{-\infty}^{\infty} e^{-ixt} f(t)dt , \qquad x \in \mathbb{R} ,$$

whenever the integral exists. L^p are the usual Lebesgue spaces with norm $\|\cdot\|_p$, while $f \in L^p_w$, w a weight, if $w^{1/p} f \in L^p$. p', the conjugate index of p, $1 < p < \infty$, is defined by $1/p + 1/p' = 1$ and similarly for q'. Finally, C denotes a constant independent of f and may be different at different occurrences.

2. Main Results. The weight functions we consider here belong to the A_p-weight class.

Definition 2.1. A non-negative locally integrable function w defined on $\mathbb{R}$ is in A_p, $1 < p < \infty$ if for every interval $I \subset \mathbb{R}$ with length $|I|$,

$$\sup_{I\subset\mathbb{R}} \left[1/|I| \int_I w(x)dx\right]^{1/p} \left[1/|I| \int_I w(x)^{1-p'}dx\right]^{1/p'} < \infty .$$

In the sequel we shall need the fact ([2, Prop. 3.3]) that if w is even, non-decreasing on $(0,\infty)$, then $w \in A_p$, $1 < p < \infty$, if and only if $|x|^{p'-2}w(1/x)^{1-p'} \in A_{p'}$. (R. Johnson and C. Neugebauer recently observed that this result holds for arbitrary weights).

Lemma 2.2. ([3 Lemma 1]). Let u be a non-negative function in $C_0(0,1)$, such that $\int_0^1 u(x)dx = 1$. If

$$U(z) = \int_0^1 e^{-zt}u(t)dt , \qquad \operatorname{Re} z \geq 0 , \tag{2.1}$$

then $|U(z)| \leq C(1+|z|)^{-1}$.

Theorem 2.3. ([2]). a) Let w be even, non-decreasing on $[0,\infty)$ and $f \in L^p_w$, $1 < p \leq q \leq p' < \infty$. Then

$$(2.2)\quad \left[\int_{-\infty}^{\infty} |\hat{f}(x)|^q |x|^{q/p'-1} w(1/x)^{q/p} dx\right]^{1/q} \le C\left[\int_{-\infty}^{\infty} |f(x)|^p w(x)dx\right]^{1/p},$$

if and only if $w^{q/p} \in A_{1+q/p'}$.

b) Also

$$\left[\int_{-\infty}^{\infty} |\hat{f}(x)|^{p'} w(x)^{1-p'} dx\right]^{1/p'} \le C\left[\int_{-\infty}^{\infty} |f(x)|^{q'} |x|^{q'/p-1} w(1/x)^{-q'/p} dx\right]^{1/q'}$$

if and only if $w^{1-p'} \in A_{1+p'/q}$.

With these preliminaries we can now prove our main result.

Theorem 2.4. If $f \in B_w^{p,q}$, $1 < p \le q \le p'$, where w is continuous, radial, non-decreasing on $(0,\infty)$ and $w^{q/p} \in A_{1+q/p'}$, then there exists a function F, such that $f(z) = (LF)(z)$, $\operatorname{Re} z > 0$, with

$$\int_0^{\infty} |F(t)|^q t^{q/p'-2} w(1/t)^{q/p} dt < \infty .$$

Proof. Following Genchev [3], let

$$I_p(x) = \int_{-\infty}^{\infty} |f(x+iy)|^p w(x+iy)dy , \qquad x \ge 0 ,$$

then $f \in B_w^{p,q}$ implies that $I_p(x) < \infty$, a.e. Moreover, there exists a sequence $\{x_j\}_{j=1}^{\infty}$ with $x_j \to \infty$, such that $I_p(x_j) \to 0$ as $j \to \infty$. Let $\Omega = \{x \in (0,\infty): I_p(x) < \infty\}$ and write $f_\delta(z) = f(z+\delta)$, where $\operatorname{Re} z \ge 0$ and $\delta \in \Omega$. Define

$$F_\delta(t) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{tz} f_\delta(z) U(\delta z) dz ,$$

where U is given by (2.1) and the integral is taken over the line $\operatorname{Re} z = 0$. Hölder's inequality and an application of Lemma 2.2 shows that the integral exists. If $x \in (0,\infty)$ and $x+\delta \in \Omega$, then we claim that

$$(2.3) \qquad F_\delta(t) = \frac{1}{2\pi i}\int_{x-i\infty}^{x+i\infty} e^{tz} f_\delta(z)U(\delta z)dz \ .$$

First by Fubini's theorem and Hölder's inequality with $w_\delta(a) = w(a+\delta)$

$$\int_{-\infty}^{\infty}\int_0^x |f_\delta(\xi+iy)|^p\, w_\delta(\xi+iy)d\xi dy \le \left\{\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty}|f_\delta(\xi+iy)|^p\, w_\delta(\xi+iy)dy\right]^{q/p} d\xi\right\}^{p/q} x^{1-p/q}.$$

Denoting the right side by $J(x)$, then $J(x) < \infty$ and again Fubini's theorem, Hölder's inequality applied twice and Lemma 2.2 yield

$$\int_{-\infty}^{\infty}\int_0^x |f_\delta(\xi+iy)|\ |U(\delta(\xi+iy))|d\xi dy$$

$$\le C\int_{-\infty}^{\infty}\left[\int_0^x |f_\delta(\xi+iy)|^p w_\delta(\xi+iy)d\xi\right]^{1/p}\left[\int_0^x w_\delta(\xi+iy)^{1-p'}(1+|\delta(\xi+iy)|)^{-p'}d\xi\right]^{1/p'} dy$$

$$\le CJ(x)^{1/p}\left\{\left(\int_0^x w(\xi)^{1-p'}d\xi\right)\left(\int_{-\infty}^{\infty}(1+|\delta y|)^{-p'}dy\right)\right\}^{1/p'} ,$$

which is finite since $w^{q/p} \in A_{1+q/p'}$. But now there exist sequences $\{a_k\}_{k=0}^{\infty}$, $\{b_k\}_{k=0}^{\infty}$ with $a_k \to -\infty$, $b_k \to \infty$, as $k \to \infty$, such that

$$\lim_{k\to\infty}\int_0^x |f_\delta(\xi+ia_k)|\ |U(\delta(\xi+ia_k))|d\xi = 0 ,$$

and the same limit holds with a_k replaced by b_k. Now integrate the holomorphic function $z \to e^{tz}f_\delta(z)U(\delta z)$ over the rectangle $T\colon 0 \le \xi \le x$, $a_k \le y \le b_k$ then as $k \to \infty$, (2.3) follows. Applying Hölder's inequality and Lemma 2.2,

$$|F_\delta(t)| \le Ce^{tx}I_p(x+\delta)^{1/p}\left\{\int_0^1 w(y)^{1-p'}dy + w(1)^{1-p'}\int_1^{\infty}(1+|\delta y|)^{-p'}dy\right\}^{1/p'} ,$$

and since with $x = x_j$, $I_p(x_j+\delta) \to 0$ as $j \to \infty$, we obtain that $F_\delta(t) = 0$ for $t \le 0$. Since

$$(2.4) \qquad F_\delta(t)e^{-tx} = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ity} f_\delta(x+iy)U(\delta(x+iy))dy ,$$

Theorem 2.3 a) shows that

$$\int_0^\infty |F_\delta(t)|^q \, e^{-txq} \, t^{q/p'-1} \, w(1/t)^{q/p} dt$$

$$\le C\left\{\int_{-\infty}^{\infty} |f_\delta(x+iy)|^p w_\delta(x+iy)dy\right\}^{q/p} ,$$

since w is non-decreasing radially. Integrating this inequality shows that $t^{1/p'-2/q} w(1/t)^{1/p} F_\delta(t) \doteq H_\delta(t) \in L^q(0,\infty)$. By weak compactness, there is a subsequence $\{\delta_k\}_{k=1}^\infty$ with $\delta_k \to 0$ as $k \to \infty$, and a function F, such that for every $G \in L^{q'}(0,\infty)$

$$\lim_{k\to\infty} \int_0^\infty H_{\delta_k}(t)G(t)dt = \int_0^\infty t^{1/p'-2/q} \, w(1/t)^{1/p} \, F(t)G(t)dt .$$

Let $G(t) = e^{-t(x+iy)} \, t^{2/q-1/p'} \, w(1/t)^{-1/p}$, then inverting (2.4) we get

$$f(x+iy) = \lim_{k\to\infty} f_{\delta_k}(x+iy)U(\delta_k(x+iy)) = \lim_{k\to\infty}\int_0^\infty H_{\delta_k}(t)G(t)dt$$

$$= \int_0^\infty e^{-t(x+iy)} F(t)dt .$$

This proves the theorem provided we justify the Fourier inversion of (2.4) and show that $G \in L^{q'}(0,\infty)$. To do this we apply Hölder's inequality, the remark following Definition 2.1 and [4, Lemma 1]. We omit the details.

Theorem 2.5. Suppose $f(z) = (LF)(z)$, $\operatorname{Re} z > 0$, where

$$\int_0^\infty |F(t)|^{q'} t^{q'/p-2} w(1/t)^{-q'/p} dt < \infty ,$$

$1 < p \le q \le p'$, w non-decreasing (as a radial function) on $(0,\infty)$ and $w^{1-p'} \in A_{1+p'/q}$, then $f \in B^{p',q'}_{w^{1-p'}}$.

The proof follows from Theorem 2.3 b), the fact that $w^{1-p'} \in A_{1+p'/q}$ if and only if $w^{q/p} \in A_{1+q/p'}$, and the remark following Definition 2.1.

Observe that if $q = p'$ and $w \equiv 1$ in Theorem 2.4 we obtain a result of Genchev [3] while the case $q = p$ yields:

Corollary 2.6. If $f \in B^{p,p}_w$, $1 < p \le 2$, and $w \in A_p$ is continuous, radial and non-decreasing on $(0,\infty)$, then there exists a function F such that $f(z) = (LF)(z)$, $\operatorname{Re} z > 0$ with $t^{1-3/p} w(1/t)^{1/p} F(t) \in L^p$

If $p = q = 2$ and $w \equiv 1$, Theorems 2.4 and 2.5 reduce to the Paley-Wiener characterization of functions in $B^{2,2}$, given by Genchev [3].

Remark. Instead of Theorem 2.3 one could apply [1, Theorem 1.1] to obtain similar results with different weights and an extended range of indices. However, in this case the weights do not belong to the A_p-class.

References

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