CONSTRUCTIVE THEORY OF FUNCTIONS'87

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ON THE DERIVATIVES OF POSITIVE LINEAR OPERATORS

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1. Introduction. Let us introduce some notations. Let us denote the Szasz, Post-Widder and Gamma operators by S_n, P_n and G_n resp. i.e.

$$(1.1)\quad S_n(f,x) = \sum_{k=\emptyset}^{\infty} f(k/n)s_{n,k}(x), \qquad s_{n,k}(x) = \exp(-nx)(nx)^k/k!,$$

$$(1.2)\quad P_n(f,n) = \int_{\emptyset}^{\infty} f(t)p_n(x,t)dt,$$

$$p_n(x,t) = \exp(-nt/x)n^n x^{-n} t^{n-1}/(n-1)!,$$

$$(1.3)\quad G_n(f,x) = \int_{\emptyset}^{\infty} f(n/t)g_n(x,t)dt, \qquad g_n(x,t) = x^{n+1}t^n\exp(-xt)/n!$$

All of the operators above are positive and linear. Let us denote by C the space of continuous and bounded functions on the interval $[\emptyset,\infty]$ with the usual supremum norm $||\cdot||$ and let C^r be equal to $\{f:f(r)\in C\}$ where r is a fixed positive integer.
Let D and I denote the operators of derivation and integration i.e.

$$Df(x) = \frac{df(x)}{dx}, \qquad If(x) = \int_{\emptyset}^{x} f(t)dt.$$

Our problem is the following: What can we say about the convergence of

$$(1.4)\qquad ||D^rL_n(f) - D^rf|| \qquad (f\in C^r)$$

where L_n denotes one of the above mentioned positive linear operators (1.1) - (1.3)?
(About the pointwise convergence see e.g. [2].) There is a widespread theory about the positive linear operators, but $D^r L_n$ is not positive. So let us introduce the following operator

$$L_{n,r}(f,x) = D^r L_n(I^r f,x)$$

Obviously the investigation of (1.4) is equivalent to the investigation of

$$(1.5) \qquad L_{n,r}(f) - f \quad (f \in C).$$

So in the following we shall deal with the convergence of (1.5).
We shall need the following modulus:

$$w_F(f,d) = \sup\{|\Delta^2_{hF(x)} f(x)| : x \geq \emptyset,\ h < d\}$$

About this modulus see [3]. In the following F will be defined as

$$F(x) = L_1((t-x)^2,x)^{1/2}$$

2. Results. Our first statement is the following:

Proposition. $L_{n,r}$ is a positive linear operator.

The difficulty in the investigation of (1.7) is due to the fact that while $L_n(f_1) = f_1$, the functions $L_{n,r}(f_1)$ are never equal to f_1 where $f_1(x) = x$.

Totik proved [3] in a more general setting that the following estimation is valid.

Theorem A: If $f \in C$ then

$$\|L_n(f) - f\| = O(w_F(f,1/\sqrt{n})).$$

Theorem B: If $f \in C$ and $\emptyset < \alpha < 1$ then

$$\text{(i)} \quad L_n(f) - f = o(1) \ (n \to \infty) \iff w(d) = o(1) \ (d \to \emptyset)$$

(ii) $L_n(f) - f = O(n^{-\alpha})$ $(n \to \infty)$ $\Longleftrightarrow$ $w(d) = O(d^{-2\alpha})$ $(d \to 0)$.

We prove the analogon of Theorem A and B for the operators $P_{n,r}$ and $G_{n,r}$.

Theorem 2.1: Let L_n be either P_n or G_n, $f \in C[0,\infty)$ and $0 < \alpha < 1$. Then we have

(i) $\|L_{n,r}(f) - f\| = O(w(f,1/\sqrt{n}) + \|f\|/n)$

(ii) $L_{n,r}(f) - f = o(1)$ $(n \to \infty)$ $\Longleftrightarrow$ $w(d) = o(1)$ $(d \to 0)$

(iii) $L_{n,r}(f) - f = O(n^{-\alpha})$ $(n \to \infty)$ $\Longleftrightarrow$ $w(d) = O(d^{2\alpha})$ $(d \to 0)$

For the Szasz-operator we prove the following

Theorem 2.2: If $f \in C$ then

(i) $S_{n,r}(f) - f = o(1)$ $(n \to \infty)$ $\Longleftrightarrow$ $w(d) = o(1)$ $(d \to 0)$

(ii) (1) $w(d) = O(d^{2\alpha})$, (2) $f(0) - f(h) = O(h^{\alpha})$ $\Longrightarrow$

$S_{n,r}(f) - f = O(n^{\alpha})$.

Remark. For the case $r = 1$ Theorem 2.2 was proved by Totik in [4]. He verified that the condition (ii) (2) can not be omitted. Because the proof is identical to the case $r = 1$ we omit it.

3. Proofs. An easy computation gives us that

(3.1) $$S_{n,r}(f,x) = \sum_{k=0}^{\infty} (n^r \Delta^r_{1/n} I^r f)(k/n)\, s_{n,k}(x),$$

(3.2) $$P_{n,r}(f,x) = \int_0^{\infty} f(t)(t/x)^r\, p_n(x,t)\,dt,$$

$$(3.3) \qquad G_{n,r}(f,x) = \frac{n^r(n-r)!}{n!}\int_{\emptyset}^{\infty} f(n/t)g_{n-r}(x,t)dt$$

which proves our Proposition.

Proof of Theorem 2.1. The proof for P_n and G_n is completely analogous. We shall prove it only for P_n. The proof is a modification of the proof of Theorem A. Let $J(x) = (3x/4, 5/4)$ and let $k(t) = k(t,x)$ be the characteristic function of $J(x)$ i.e.

$$k(t,x) = \begin{cases} 1 & \text{if} \quad t \in J(x), \\ \emptyset & \text{otherwise.} \end{cases}$$

The proof of Theorem A is based on the following Lemma 3.1. It is enough to verify the lemma, the other parts of the proof are identical.

Lemma 3.1. If $x > \emptyset$, $g \in C(J(x)) \cap C((\emptyset, \ )-J(x))$ and g is convex on $J(x)$ then

$$(3.4) \qquad L_{n,r}(g,x) \geq g(x) - K\|g\|/n$$

where $K > \emptyset$ is an absolute constant.

Proof: We shall use the well-known relation

$$(3.5) \qquad \int_a^b t^k \exp(-ut)dt = \Big[- k!\, u^{-k-1}\exp(-ut)\sum_{j=\emptyset}^{k}\frac{(ut)^j}{j!}\Big]_a^b$$

Using (1.2), (3.2) and (3.5) we have by direct computation

$$(3.6) \qquad P_{n,r}(f_j,x) = x^j\,\frac{(n+r+j-1)!}{n^{r+j}\,(n-1)!} = x^j(1+O(1/n))$$

where $f_j(x)=x^j$. From the convexity of g on J(x) it follows the existence of a constant d so that

$$(3.7) \qquad g(t) \geq g(x) + d(t-x) \qquad (t \in J(x)),$$

i.e.

(3.8) $$g(t)k(t) \geq g(x)k(t) + d(t-x)k(t) \qquad (t > \emptyset).$$

So

$$P_{n,r}(gk,x) \geq g(x)P_{n,r}(k,x) + dP_{n,r}((t-x)k,x),$$

from where

(3.9) $$P_{n,r}(g,x) \geq g(x) - g(x)[1-P_{n,r}(k,x)] - P_{n,r}((k-1)g,x) + dP_{n,r}((t-x)k,x).$$

From (3.7) and the definition of $J(x)$ we can write

(3.1∅) $$2\|g\| \geq g(5x/4)-g(x) \geq dx/4 \geq g(x)-g(3x/4) \geq -2\|g\|.$$

If we can verify that

(3.11) $$\|P_{n,r}(k-1)\| = O(1/n),$$

(3.12) $$\|P_{n,r}(k)-1\| = O(1/n),$$

(3.13) $$\|P_{n,r}((t-x)k)/x\| = O(1/n),$$

then from (3.9) and (3.1∅) the statement of the lemma already follows. By (3.2) and (3.6) the relations (3.11) and (3.12) are equivalent so we have to prove (3.11) and (3.13) only. During the computations we shall use the relation

$$\exp(-x)\sum_{k=\emptyset}^{\infty} x^k/k! = 1$$

without further reference.

$$P_{n,r}(1-k,x) = \frac{n^n}{(n-1)!} x^{-n-r} \int_{[\emptyset,\infty)-J(x)} t^{n+r-1}\exp(-nt/x)dt =$$

$$= \frac{n^n}{(n-1)!} x^{-n-r}\{ \int_{\emptyset}^{3x/4} + \int_{5x/4}^{\infty} \} = A + B.$$

Applying (3.5) we have

$$A =$$

$$= \frac{n^n}{(n-1)!} x^{-n-r}\left[-(n+r-1)!(x/n)^{n+r}\exp(-nt/x) \sum_{j=\emptyset}^{n+r-1} \frac{(nt/x)^j}{j!}\right]_{\emptyset}^{3x/4} =$$

$$= \frac{(n+r-1)!}{n^r(n-1)!} \exp(-3n/4) \sum_{j=n+r}^{\infty} \frac{(3n/4)^j}{j!} .$$

So by [1,p.2ØØ]

$$A = O(\exp(-c1x))$$

with a suitable $c1 > \emptyset$. The estimation of B is analogous, we omit the details.

$$P_{n,r}((t-x)k,x)/x = \int_{[\emptyset,\infty)-J(x)} (t/x-1)(t/x)^r p_n(x,t)dt = \int_{\emptyset}^{3x/4} + \int_{5x/4}^{\infty} = C+D.$$

Using (3.4) and [1, p. 2ØØ] we get for both C and D the same estimation as for A, above. This proves the lemma and so the proof of Theorem 2.1 (i) is complete.

To prove (ii) we use the following

<u>Lemma 3.2.</u>

$$\|P_n - P_{n,r}\| = O(1/\sqrt{n}).$$

<u>Proof:</u> Using the definition of P_n and (3.6) we have

$$\|P_n - P_{n,r}\| = \frac{n^n}{(n-1)!} \int_{\emptyset}^{\infty} |1 - (t/x)^r| x^{-n} t^{n-1} \exp(-nt/x)dt =$$

$$= 2 \frac{n^n}{(n-1)!} \int_{\emptyset}^{x} (1-(t/x)^r) x^{-n} t^{n-1} \exp(-nt/x)dt + \frac{(n+r-1)!}{n^r(n-1)!} - 1 =$$

$$= 2 \frac{n^n}{(n-1)!} x^{-n}[\exp(-nt/x)\{-(n-1)!(x/n)^n \sum_{j=\emptyset}^{n-1} (nt/x)^j/j! +$$

$$+ x^{-r}(n+r-1)!(x/n)^{n+r} \sum_{j=\emptyset}^{n+r-1} (nt/x)^j/j!\}]_{\emptyset}^{x} + O(1/n) =$$

$$= 2(1 - \frac{(n+r-1)!}{n^r(n-1)!} e^{-n} \sum_{j=n+r}^{\infty} n^j/j! + 2e^{-n} \sum_{j=n}^{n+r-1} n^j/j! + O(1/n) =$$

$$= O(1/n) + O(e^{-n}n^n/n!) = O(1/\sqrt{n})$$

where in the last step we used the Stirling formula.

To prove (iii) we apply [3] Proposition 2. Corollary according to it is enough to prove

$$(3.14) \qquad |x^2 P''_{n,r}(f,x)| = O(\|f\|/n) \quad (f \in C)$$

and

$$(3.15) \qquad |x^2 P''_{n,r}(f,x)| = O(\|x^2 f''\| + \|f\|) \quad (f \in C,\ f' \text{ loc. abs. cont.})$$

But

$$|x^2 P''_{n,r}(f,x)| = |P_{n,r}(t^2 f''(t),x)| = O(\|t^2 f''\|)$$

and

$$x^2 P''_{n,r}(f,x) = \int_{\emptyset}^{\infty} f(t)(t/x)^r p_n(x,t)x^{-2}q(n,x)dt$$

where

$$q(n,t) = n^2(t-x)^2 - 2n(r+1)tx + x^2(n(2r+1)+r(r+1))$$

so

$$|x^2 P''_{n,r}(f,x)| \le \|f\| \int_{\emptyset}^{\infty} (t/x)^r p_n(x,t)x^{-2}Q(n,x)dt =$$

$$= 4(r+1)\|f\|(n+r)!/(n^r(n-1)!)$$

where

$$Q(n,t) = n^2(t-x)^2 + 2n(r+1)tx + x^2(n(2r+1) + r(r+1)).$$

Verifying (3.14) and (3.15) the proof is complete.

References

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