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OPERATORS FOR ONESIDED APPROXIMATION OF FUNCTIONS

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Abstract. We define nonlinear operators mapping any bounded measurable function to trigonometric polynomials or entire functions of exponential type σ. These operators realize the order of best onesided approximations.

0. Introduction. The first authors working on onesided approximation - Freud[3], Ganelius[4] - used operators for constructing polynomials of onesided approximation. These operators were defined for functions f such that $f^{(k)}$, k=0,1,..., is of bounded variation. Later on Andreev, Popov and Sendov [1,7,10] characterized the order of best onesided trigonometric approximation to bounded measurable functions in terms of the average moduli. For obtaining direct statements they construct smooth functions close to the original function and after that they approximate onesidedly the smoothing functions. As far as we know there are no simple definitions of operators for onesided approximation which works for any measurable function. This was one of the reasons leading us to investigations of operators good for onesided approximation. One difficulty here is the fact (see 5.3) that any operator for onesided approximation (exept the identity) has to be essentialy nonlinear. To compare with the case of best approximation remark that a convolution of the function with a linear combination of Jackson kernels (a linear operator!) provides polynomials of almost best approximation (see (3.8),(3.9)).

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The second reason motivating our investigation was the difficulties occuring in the attempts to transfer the method of proving the above mentioned results of Andreev, Popov and Sendov to the multivariate case. Direct and converse multivariate theorems are stated without proofs by Popov[8]. The direct theorem announced in [8] is proved in Section 3. Even in the univariate case our proof is simpler than this one given in [10].

1. Notations. We shall consider bounded measurable realvalued functions defined on Ω, where $\Omega=\mathbb{R}^d$, or $\Omega=[0,2\pi)^d$ for 2π periodic functions, or $\Omega=[0,1]^d$. We consider $\mathbb{R}^d$ as a normed vector space with elements x,y,h, $x=(x_1,x_2,\dots,x_d)$, and norm $|x|=\max\{|x_i|:i=1,2,\dots,d\}$. $U(\delta,x)=\{y\in\mathbb{R}^d: |y-x|\leq\delta\}$ is the δ neighbourhood of the point x.

α, β, ε are multyindices. $|\alpha|=\alpha_1+\alpha_2+\dots+\alpha_d$ is the length of α, where $\alpha=(\alpha_1,\alpha_2,\dots,\alpha_d)$. $\alpha\leqq\beta$ means $\alpha_i\leqq\beta_i$ for any i and $\binom{\alpha}{\beta}=\prod_{i=1}^{d}\binom{\alpha_i}{\beta_i}$. D^α as usual denotes a differential operator in R^d.

By $P_N(T_N)$ we denote the set of all algebraic (trigonometric) polynomials in R^d of <u>total</u> degree not greater than N. A_σ denotes the set of all entire functions in $\mathbb{R}^d$ of exponential type σ.

Let $\tilde{E}(X;f)_p$ be the best onesided approximation in L_p, $1\leqq p\leqq\infty$, by elements of X ($X=T_N$ or A_σ), i.e.

$$\tilde{E}(X;f)_p=\inf\{\|g^+-g^-\|_p:\ g^\pm\in X,\ g^-\leqq f\leqq g^+\}.$$

Let $\Delta_h^k f(x)$ denote the k-th finite diference with step h of f in the point x. We denote by

$$\omega_k(f,x;\delta)=\sup\{|\Delta_h^k f(y)|:\ y,y+kh\in U(k\delta/2,x)\}$$

the local modulus of f (natural k, $\delta>0$). Two global moduli of the function f will be used - the usual modulus of smoothness

$$\omega_k(f;\delta)_{p(\Omega)}=\sup\{\|\Delta_h^k f(.)\|_{p(\Omega)}:\ |h|\leq\delta\}$$

and the average modulus of smoothness

$$\tau_k(f;\delta)_{p(\Omega)}=\|\omega_k(f,.;\delta)\|_{p(\Omega)}.$$

The properties of ω_k are assumed to be known. Some properties of τ_k are given in [8].

In the paper k,d,p are fixed numbers, k,d - naturals standing for the order of the moduli and the dimention of the space respectively, $1\leqq p\leqq\infty$. By c we denote positive numbers which may depend only on k,d and p. The numbers $m=[d/p]+1$ ([.] - integral part) and $\ell=\max\{k,m\}$ are also fixed.

We introduce the onesided K-functional as the quantity

$$(1.1)\quad \tilde{K}_k(f,t^k)_p=\inf\{\|g^+-g^-\|_p+\sum_{|\alpha|=k,\ell} t^{|\alpha|}(\|D^\alpha g^+\|_p+\|D^\alpha g^-\|_p)\}$$

where the inf is taken over all $g^\pm\varepsilon W_p^\ell(\Omega)$ such that $g^-\leqq f\leqq g^+$. Let us mention that in (1.1) we have only the sum for $|\alpha|=k$ when $k>d/p$ and the sums for $|\alpha|=k$ and $|\alpha|=\ell$ when $k\leqq d/p$. The last case is of importance for the multivariate case.

2. Equivalence of the onesided K-functional and the average moduli. In this section we prove that the onesided K-functional (1.1) and the average moduli are equivalent. This result differ from Theorem 1 in Popov[15] by the definition of the K-functional. We utilize (1.1) because of forthcoming advantages (see Corollary 1). In the context of this paper we prove Theorem 1 only to apply it in Corollary 1 but the fact of equivalence is important by itself. For d=1 Theorem 1 is proved in [9].

Theorem 1. Let $0<t\leqq\operatorname{diam}\Omega$. Then

$$(2.1)\quad c\tau_k(f;t)_p\leqq \tilde{K}_k(f,t^k)_p\leqq c\tau_k(f;t)_p.$$

Proof. We begin with the proof of the second inequality in (2.1). We see from (1.1) that it is enough to constuct two functions g^+ and g^- from $W_p^\ell(\Omega)$ satisfying the conditions:

1) $g^-(x)\leqq f(x)\leqq g^+(x)$ for every $x\varepsilon\Omega$;

2) $\|g^+-g^-\|_p\leqq c\tau_k(f;t)_p$;

3) $\|D^\beta g^\pm\|_p\leqq ct^{-|\beta|}\tau_k(f;t)_p$ for every β, $k\leqq|\beta|\leqq\ell$.

Let t be fixed. We set $\delta=t/4$ when $\Omega=\mathbb{R}^d$, $N=[4/t]+1$, $\delta=1/N$ when $\Omega=[0,1]^d$ and $N=[8\pi/t]+1$, $\delta=2\pi/N$ when $\Omega=[0,2\pi)^d$. Denote

$$Z(\Omega)=\begin{cases} Z^d=\{0,\pm1,\pm2,\dots\}^d, & \Omega=\mathbb{R}^d;\\ \{0,1,2,\dots,N\}^d, & \Omega=[0,1]^d;\\ \{0,1,2,\dots,N-1\}^d, & \Omega=[0,2\pi)^d.\end{cases}$$

For any $\mathbf{j}=(j_1,\dots,j_d)\varepsilon Z^d$ we consider the following cubs in $\mathbb{R}^d$.

$$(2.2)\quad \Omega_{\mathbf{j}}=\{x\varepsilon\mathbb{R}^d:\ |x-\mathbf{j}\delta|\leqq\delta\},$$

$$\Omega'_{\mathbf{j}}=\{x\varepsilon\Omega_{\mathbf{j}}:\ x_i\geqq\delta j_i\}.$$

Let μ *) be a fixed $C^{\infty}(\mathbb{R})$ function such that $\mu(x)=0$ for $x\leqq 0$, $\mu(x)=1$ for $x\geqq 1$, $0<\mu(x)<1$ for $0<x<1$. For any $\Omega_{\mathbf{j}}$ we set

$$\mu_{\mathbf{j}}(x) = \prod_{i=1}^{d} \mu(x_i/\delta - j_i+1)(1-\mu(x_i/\delta - j_i)).$$

Properties following immediately from the definition:

(2.3) $0\leqq \mu_{\mathbf{j}}(x)\leqq 1$ for any $x\varepsilon R^d$; $\mu_{\mathbf{j}}(\mathbf{j}\delta)=1$;

(2.4) $\mu_{\mathbf{j}}(x)=0$, $x\bar{\varepsilon}\,\Omega_{\mathbf{j}}$;

(2.5) $\sum_{\mathbf{j}\varepsilon Z^d} \mu_{\mathbf{j}}(x) = 1$ for any $x\varepsilon \mathbb{R}^d$.

For $\mathbf{j}\,\varepsilon\, Z(\Omega)$ we set

(2.6) $e_{\mathbf{j}}(f)=\inf\{\|f-R\|_{\infty(\Omega_{\mathbf{j}})} : R\varepsilon P_{k-1}\} = \|f-R_{\mathbf{j}}\|_{\infty(\Omega_{\mathbf{j}})}$

where $R_{\mathbf{j}}\,\varepsilon\, P_{k-1}$ (Remark. For the case $\Omega=[0,1]^d$ and $\mathbf{j}$ such that $\mathbf{j}\delta$ is on the boundary of Ω we replace $\Omega_{\mathbf{j}}$ by $\Omega_{\mathbf{j}}\cap\Omega$ in (2.6).).

From the Whitney's theorem (see e.g. [6]) we have

$e_{\mathbf{j}}(f) \leqq c \sup\{|\Delta_h^k f(x)| : x, x+kh\,\varepsilon\Omega_{\mathbf{j}}\}$ and hence

(2.7) $e_{\mathbf{j}}(f) \leqq c\,\omega_k(f,x;t)$ for any $x\,\varepsilon\Omega_{\mathbf{j}}$.

We set $R^{\pm}_{\mathbf{j}}(x) = R_{\mathbf{j}}(x)\pm e_{\mathbf{j}}(f)$. For any $x\,\varepsilon\Omega_{\mathbf{j}}$ we have

(2.8) $R^{-}_{\mathbf{j}}(x) \leqq f(x) \leqq R^{+}_{\mathbf{j}}(x)$.

Finaly we define

(2.9) $g^{\pm}(x) = \sum_{\mathbf{j}\,\varepsilon Z(\Omega)} \mu_{\mathbf{j}}(x) R^{\pm}_{\mathbf{j}}(x)\ \varepsilon\ C^{\infty}(\mathbb{R}^d)$.

Now 1) follows from (2.9),(2.8) and (2.2) - (2.6). From (2.9) we get $g^{+}(x)-g^{-}(x) = 2\sum_{\mathbf{j}\varepsilon Z(\Omega)} \mu_{\mathbf{j}}(x)\, e_{\mathbf{j}}(f)$ and hence

$0\leqq g^{+}(x)-g^{-}(x) \leqq c\,\omega_k(f,x;t)$ by (2.7), (2.3) and (2.5). Now 2) follows from the above inequality. Take β, $k\leqq|\beta|\leqq\ell$. Let $x\varepsilon\Omega'_{\mathbf{j}}$. Then from (2.4), (2.5) and (2.9) we get

$$g^{+}(x) = R^{+}_{\mathbf{j}}(x) + \sum_{\varepsilon_i=0,1} \mu_{\mathbf{j}+\boldsymbol{\varepsilon}}(x)(R^{+}_{\mathbf{j}+\boldsymbol{\varepsilon}}(x)-R^{+}_{\mathbf{j}}(x))$$

and therefore

$$D^{\beta}g^{+}(x)=D^{\beta}R^{+}_{\mathbf{j}}(x)+\sum_{\varepsilon_i=0,1}\ \sum_{0\leqq\alpha\leqq\beta}\binom{\beta}{\alpha}D^{\beta-\alpha}\mu_{\mathbf{j}+\boldsymbol{\varepsilon}}(x).D^{\alpha}(R^{+}_{\mathbf{j}+\boldsymbol{\varepsilon}}(x)-R^{+}_{\mathbf{j}}(x)).$$

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*) e.g. $\mu(x)=\int_0^x\varphi \Big/ \int_0^1\varphi$ for $0<x<1$, where $\varphi(x)=\exp(-1/(x-x^2))$.

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Having in mind the definitions of R_j and μ_j, (2.7) and Markov's inequality we get

$$\|D^\beta g^+\|_{p(\Omega_j')} \leqq c \sum_{\varepsilon_i=0,1} \sum_{0\leqq\alpha\leqq\beta} \|D^{\beta-\alpha}\mu_{j+\varepsilon}\|_{\infty(\Omega_j')} \|D^\alpha(R_{j+\varepsilon}^+ - R_j^+)\|_{p(\Omega_j')}$$

$$\leqq c \sum_{\varepsilon_i=0,1} \sum_{0\leqq\alpha\leqq\beta} t^{-|\beta-\alpha|}.t^{-|\alpha|} \|R_{j+\varepsilon}^+ - R_j^+\|_{p(\Omega_j')}$$

$$\leqq c\, t^{-|\beta|} \sum_{\varepsilon_i=0,1} \{\|R_{j+\varepsilon}^+ - f\|_{p(\Omega_j')} + \|R_j^+ - f\|_{p(\Omega_j')}\}$$

$$\leqq c\, t^{-|\beta|} \|\omega_k(f,.;t)\|_{p(\Omega_j')}.$$

Summating on $j\varepsilon Z(\Omega)$ the above inequality we get 3). This completes the proof of the second inequality in (2.1).

We continue with the proof of the first inequality in (2.1). The following imbedding theorem (see [11, Theorems 18.10, 18.11, pp.302, 303]) will play the main role in this part.

Let $\Omega=[0,1]^d$, $g\varepsilon L_{p(\Omega)}$, $D^\alpha g\varepsilon L_{p(\Omega)}$ (generalized derivatives) for any α, $|\alpha|=\ell$ (recall $\ell>d/p$). Then g is equivalent to $G\varepsilon C(\Omega)$ and

(2.10) $$\|G\|_{\infty(\Omega)} \leqq c \{\|g\|_{p(\Omega)} + \sum_{|\alpha|=\ell} \|D^\alpha g\|_{p(\Omega)}\}.$$

Making a linear change of the variables in (2.10) we get for any $x\varepsilon\Omega$ and $\delta\leqq 1$

(2.11) $$\|G\|_{\infty(U(\delta,x)\cap\Omega)} \leqq c\,\delta^{-d/p} \{\|g\|_{p(U(\delta,x)\cap\Omega)} + \delta^\ell \sum_{|\alpha|=\ell} \|D^\alpha g\|_{p(U(\delta,x)\cap\Omega)}\}.$$

Let $g^\pm \varepsilon W_p^\ell(\Omega)$, $g^-\leqq f\leqq g^+$. We have

(2.12) $$\tau_k(f;t)_{p(\Omega)} \leqq \tau_k(f-g^-;t)_{p(\Omega)} + \tau_k(g^-;t)_{p(\Omega)}.$$

Using (2.11) with $G=g=g^+-g^-$ we have

(2.13) $$\omega_k(f-g^-,x;t) \leqq 2^k \|f-g^-\|_{\infty(U(kt/2,x))} \leqq 2^k \|g^+-g^-\|_{\infty(U(kt/2,x))}$$

$$\leqq c\, t^{-d/p} \{\|g^+-g^-\|_{p(U(kt/2,x))} + t^\ell \sum_{|\alpha|=\ell} \|D^\alpha(g^+-g^-)\|_{p(U(kt/2,x))}\}.$$

Noticing that

(2.14) $$t^{-d/p} \| \|G\|_{p(U(kt/2,.))} \|_{p(\Omega)} \leqq k^{d/p} \|G\|_{p(\Omega)}$$

for any $G\varepsilon L_{p(\Omega)}$, from (2.13) we get

$$(2.15)\ \tau_k(f-g^-;t)_p \leqq c\{\|g^+-g^-\|_{p(\Omega)}+t^{|\alpha|}\sum_{|\alpha|=\ell}(\|D^\alpha g^+\|_p+\|D^\alpha g^-\|_p)\}.$$

For estimating the second term in the righthand side of (2.12) we consider two cases.

a) $k>d/p$, i.e. $k\geqq m$. From Theorem 1 in [5] we get

$$(2.16)\ \tau_k(g^-;t)_p \leqq ct^{d/p}\int_0^t \omega_k(g^-;u)_p u^{-d/p-1}du$$

$$\leqq ct^{d/p}\int_0^t \sum_{|\alpha|=k}\|D^\alpha g^-\|_p u^{k-d/p-1}du = ct^k\sum_{|\alpha|=k}\|D^\alpha g^-\|_p.$$

b) $k\leqq d/p$, i.e. $k<m=\ell$. From a generalization of Whitney theorem (see [6]) there is $R\in P_{k-1}$ such that

$$\|g^--R\|_{p(U(kt/2,x))} \leqq c\,\omega_k(g^-;t)_{p(U(kt/2,x))}.$$

From this inequality and (2.11) we get

$$\omega_k(g^-,x;t)=\omega_k(g^--R,x;t) \leqq 2^k\|g^--R\|_{\infty(U(kt/2,x))}$$

$$\leqq ct^{-d/p}\{\|g^--R\|_{p(U(kt/2,x))}+t^m\sum_{|\alpha|=m}\|D^\alpha(g^--R)\|_{p(U(kt/2,x))}\}$$

$$\leqq ct^{-d/p}\{\omega_k(g^-;t)_{p(U(kt/2,x))}+t^m\sum_{|\alpha|=m}\|D^\alpha g^-\|_{p(U(kt/2,x))}\}.$$

Taking L_p norm with respect to x in the above inequality and using (2.14) we get

$$(2.17)\ \tau_k(g^-;t)_{p(\Omega)} \leqq c\{\omega_k(g^-;t)_{p(\Omega)}+t^m\sum_{|\alpha|=m}\|D^\alpha g^-\|_{p(\Omega)}\}$$

$$\leqq c\{t^k\sum_{|\beta|=k}\|D^\beta g^-\|_{p(\Omega)}+t^m\sum_{|\alpha|=m}\|D^\alpha g^-\|_{p(\Omega)}\}.$$

From (2.12), (2.15), (2.16) and (2.17) we obtain

$$\tau_k(f;t)_p \leqq c\{\|g^+-g^-\|_{p(\Omega)}+\sum_{|\alpha|=k,\ell}t^{|\alpha|}(\|D^\alpha g^+\|_{p(\Omega)}+\|D^\alpha g^-\|_{p(\Omega)})\}.$$

Taking infimum on $g^\pm\in W_p^\ell(\Omega)$, $g^-\leqq f\leqq g^+$, in the above inequality we complete the proof of (2.1).

One important consequence of Theorem 1 is

Corollary 1. Let $\lambda>1$, $0<\lambda t\leqq \operatorname{diam}\Omega$. Then

$$\tau_k(f;\lambda t)_{p(\Omega)} \leqq c\,\lambda^\ell\,\tau_k(f;t)_{p(\Omega)}.$$

The proof follows immediately from (2.1) and the obvious inequality $\tilde{K}_k(f,(\lambda t)^k)_p \leqq \lambda^\ell\tilde{K}_k(f,t^k)_p$.

3. Operators for onesided approximation by trigonometric polynomials. In this section we consider the 2π periodic case, i.e. $\Omega=[0,2\pi)^d$. Let $F_n(v)=(\sin^2\pi/2n.\sin^2 nv/2)/\sin^2 v/2 \in T_{n-1}$ be Fejer kernel (normalized in an appropriate way). We shall use the following properties of F_n

(3.1) $F_n(v) \geqq 0$ for any $x \in \mathbb{R}$;

(3.2) $F_n(v) \geqq 1$ for $|v| \leqq \pi/n$;

(3.3) $\int_{-\pi}^{\pi} F_n(v)dv = 2\pi n\sin^2\pi/2n \leqq c/n$ (see [13, vol. 1, p.147]);

(3.4) $\sum_{i=0}^{n-1} F_n(v-2i\pi/n) = (n\sin\pi/2n)^2 \leqq c$ (see e.g. [13,vol.2,p.36]).

For $x=(x_1,x_2,\ldots,x_d) \in \mathbb{R}^d$ we set

(3.5) $$\Phi_n(x) = \prod_{i=1}^{d} F_n(x_i) \in T_{d(n-1)}.$$

Set $Z(\Omega) = \{0,1,\ldots,n-1\}^d$.

Lemma 1. For any $\{a_{\mathbf{j}}\}_{\mathbf{j}\in Z(\Omega)}$, $a_{\mathbf{j}} \geqq 0$, we have

$$\|\sum_{\mathbf{j}\in Z(\Omega)} a_{\mathbf{j}}\Phi_n(\cdot-2\pi\mathbf{j}/n)\|_{p(\Omega)} \leqq c((2\pi/n)^d \sum_{\mathbf{j}\in Z(\Omega)} a_{\mathbf{j}}^p)^{1/p}.$$

Proof. From (3.4),(3.5) and Jensen inequality we have

$$|\sum_{\mathbf{j}\in Z(\Omega)} a_{\mathbf{j}}\Phi_n(x-2\pi\mathbf{j}/n)|^p \leqq (n\sin\pi/2n)^{2d(p-1)} \sum_{\mathbf{j}\in Z(\Omega)} a_{\mathbf{j}}^p\Phi_n(x-2\pi\mathbf{j}/n),$$

which together with (3.3) and (3.5) proves the lemma.

We shall also use Jackson kernels

$$J_{\ell,n}(v) = \gamma_{\ell,n}((\sin nv/2)/\sin v/2)^{2\ell+2} \in T_{(\ell+1)(n-1)},$$

where $\gamma_{\ell,n}$ is chosen so that $\int_{-\pi}^{\pi} J_{\ell,n}(v)dv=1$. The following property of Jackson kernels is well known (see e.g. [14, p.193])

(3.6) $\int_{-\pi}^{\pi} J_{\ell,n}(v)|v|^r dv \leqq cn^{-r}$ for $r=0,1,\ldots,2\ell$.

For $x\in\mathbb{R}^d$ we set

(3.7) $$I_{\ell,n}(x)= J_{\ell,n}(x_1)J_{\ell,n}(x_2)\ldots J_{\ell,n}(x_d) \in T_{d(\ell+1)(n-1)}.$$

Consider the operator

(3.8) $$Q_n(f;x)= \int_\Omega \sum_{r=1}^{k} (-1)^{r+1}\binom{k}{r} f(x+ry)I_{\ell,n}(y)dy.$$

$Q_n f \varepsilon T_{d(\ell+1)(n-1)}$ and $Q_n f$ is a polynomial which realize the order of the best approximation of f (see e.g. Nikolskii [14,pp.190-195]), i.e.

(3.9) $$\|f-Q_n f\|_{p(\Omega)} \leqq c\,\omega_k(f,n^{-1})_{p(\Omega)} .$$

For the validity of (3.9) it is enough to choose in (3.8) any $\ell \geqq k$, but for our next purposes we need $\ell=\max\{k,m\}$.

Using Q_n we construct our onesided operators as follows:

(3.10) $$Q_n^{\pm}(f;x)=Q_n(f,x)\pm \sum_{j\varepsilon Z(\Omega)} \Phi_n(x-\frac{2\pi}{n}j).\ \sup_{|y-\frac{2\pi}{n}j|\leqq\frac{\pi}{n}} \{|f(y)-Q_n(f;y)|\}.$$

Theorem 2. For any bounded and measurable in Ω function f we have

(3.11) $$Q_n^{\pm}f \ \varepsilon\ T_{d(\ell+1)(n-1)};$$

(3.12) $$Q_n^{-}(f;x) \leqq f(x) \leqq Q_n^{+}(f;x) \quad \text{for any } x\varepsilon\Omega ;$$

(3.13) $$\|Q_n^{+}f-Q_n^{-}f\|_{p(\Omega)} \leqq c\,\tau_k(f;n^{-1})_{p(\Omega)} .$$

Proof. (3.11) follows from (3.10), (3.5) and (3.8). From (3.2) and (3.5) we have $\Phi_n(x)\geqq 1$ for $|x|\leqq\pi/n$, which together with the positivity of Φ_n gives (3.12). From (3.8) we have

$$\sup_{|y-\frac{2\pi}{n}j|\leqq\frac{\pi}{n}} |f(y)-Q_n(f;y)|\leqq \sup_{|y-\frac{2\pi}{n}j|\leqq\frac{\pi}{n}} \left|\int_\Omega \Delta_z^k f(y) I_{\ell,n}(z)dz\right|$$

$$\leqq\int_\Omega \sup_{|y-\frac{2\pi}{n}j|\leqq\frac{\pi}{n}} |\Delta_z^k f(y)|\, I_{\ell,n}(z)dz \leqq\int_\Omega \omega_k(f,2\pi j/n;\pi/n+|z|) I_{\ell,n}(z)dz.$$

Now using Lemma 1 with $a_j=\int_\Omega \omega_k(f,\frac{2\pi}{n}j;\frac{\pi}{n}+|z|)I_{\ell,n}(z)dz$,

Minkovski's inequality, Corollary 1, (3.7) and (3.6) we get

$$\|Q_n^{+}f-Q_n^{-}f\|_{p(\Omega)}=2\Big\|\sum_{j\varepsilon Z(\Omega)} \Phi_n(.-2\pi j/n).\ \sup_{|y-\frac{2\pi}{n}j|\leqq\frac{\pi}{n}} |f(y)-Q_n(f;y)|\Big\|_{p(\Omega)}$$

$$\leqq 2\Big\|\sum_{j\varepsilon Z(\Omega)} \Phi_n(.-2\pi j/n)a_j\Big\|_{p(\Omega)}\leqq c\Big((2\pi/n)^d \sum_{j\varepsilon Z(\Omega)} a_j^p\Big)^{1/p}$$

$$\leqq c\Big(\sum_{j\varepsilon Z(\Omega)} \int_{U(\frac{\pi}{n},\frac{2\pi}{n}j)} \Big(\int_\Omega \omega_k(f,x;3\pi/n+|z|)I_{\ell,n}(z)dz\Big)^p dx\Big)^{1/p}$$

$$= c\Big(\int_\Omega\Big(\int_\Omega \omega_k(f,x;3\pi/n+|z|)I_{\ell,n}(z)dz\Big)^p dx\Big)^{1/p}$$

$$\leqq c\int_\Omega\Big(\int_\Omega \omega_k(f,x;3\pi/n+|z|)^p dx\Big)^{1/p} I_{\ell,n}(z)dz=c\int_\Omega \tau_k(f;3\pi/n+|z|)_p I_{\ell,n}(z)dz$$

$\leqq c \int_{\Omega} (3\pi+n|z|)^{\ell} I_{\ell,n}(z)dz . \tau_k(f;\frac{1}{n})_p = c\, \tau_k(f;\frac{1}{n})_p .$

Theorem 3. $\tilde{E}(T_N,f)_p \leqq c\tau_k(f;1/N)_p$.

Proof. From Theorem 2 with $n=[N/(d(\ell+1))]+1$ and Corollary 1 we get $\tilde{E}(T_N,f)_p \leqq \|Q_n^+f-Q_n^-f\|_p \leqq c\tau_k(f;1/n)_p \leqq c\tau_k(f;d(\ell+1)/N)_p \leqq c\tau_k(f;1/N)_p$.

Theorem 3 is proved for d=1 in [10] and for d=2 by L.Alexandrov (a private communication). In [7,8] the following converse theorem is given

$$(3.14) \quad \tau_k(f;1/N)_p \leqq \frac{c}{N^k} \sum_{s=0}^{N} (s+1)^{k-1}\tilde{E}(T_s,f)_p .$$

Combining Theorem 3 with (3.14) we get

Corollary 2. Let $0<\rho<k$. Then

$\tilde{E}(T_n,f)_p = O(N^{-\rho}) \quad \text{iff} \quad \tau_k(f;t)_p = O(t^{\rho})$.

4. Onesided approximation by entire functions of exponential type. In this section we obtain results for entire functions parallel to those from section 3. Now $\Omega=\mathbb{R}^d$, $Z(\Omega)=Z^d$. We set

$F_\sigma(v) = (\pi/\sigma)^2(\sin(\sigma v/2)/v)^2 \in A_\sigma$ and

$J_{\ell,\sigma}(v) = \gamma_{\ell,\sigma}(\sin(\sigma v/2)/v)^{2\ell+2} \in A_{(\ell+1)\sigma}$.

Now Φ_σ, $I_{\ell,\sigma}$, Q_σ and $Q_\sigma^{\pm}$ are given by (3.5), (3.7), (3.8) and (3.10) respectively (n replaced by σ).

Theorem 4. Let $f\in L_p(\Omega)$, f be bounded and $\tau_1(f;1)<\infty$. Then $Q_\sigma^{\pm}f\in A_{d(\ell+1)\sigma}$, $Q_\sigma^-(f;x) \leqq f(x) \leqq Q_\sigma^+(f;x)$ for any $x\in\mathbb{R}^d$ and

$\|Q_\sigma^+f-Q_\sigma^-f\|_{p(\Omega)} \leqq c\, \tau_k(f;\sigma^{-1})_p$.

The proof follows along the lines of the proof of Theorem 2.

Theorem 4 is announced by Dryanov in [2] and proved for d=1 and d=2 in [12].

A converse type inequality for $\tilde{E}(A_\sigma,f)_p$ is proved in [12] and therefore an equivalence similar to that one in Corollary 2 is also valid.

5. Remarks and open problems.

5.1. When $\Omega=\mathbb{R}^d$ or $\Omega=[0,2\pi)^d$ one can replace $g^{\pm}$ from (2.9) (the proof of Theorem 1) by

$$g^{\pm}(x) = (t/2)^{-\ell d} \int_{U(t/4,0)} \dots \int_{U(t/4,0)} \{\sum_{j=1}^{\ell} (-1)^{j+1}\binom{\ell}{j} f(x+jz/\ell)$$

$$\pm\, \omega_{\ell}(f,x+z;t)\}\, dh^1 \dots dh^{\ell},$$

where $z=h^1+\dots+h^{\ell}$, $h^i \in U(t/4,0)\subset\mathbb{R}^d$. We utilize (2.9) because the above definition has to be essentially modified for $\Omega=[0,1]^d$.

5.2. The onesided K-functionals in (1.1) and [15] are equivalent virtue of Theorem 1 and Theorem 1 in [15]. Both K-functionals are of the form $\inf\{\|g^+-g^-\|+\sum_{\alpha\in A} t^{|\alpha|}(\|D^\alpha g^+\|+\|D^\alpha g^-\|)\}$.

A natural question when $k\leqq d/p$ is: How small (in some sense) can A be taken, preserving $|\alpha|\geqq k$, so that the new onesided K-functional will be still equivalent to $\tau_k(f;t)_p$? For example if $A=\{\alpha : k\leqq|\alpha|\leqq r, 0\leqq \alpha_i\leqq k\}$ how small can r be chosen? The arguments of section 2 show that any imbedding theorem replacing (2.10) can produce a corresponding onesided K-functional, equivalent to τ_k.

5.3. Let X be a centre symmetric body of realvalued functions. Let $Q:X\to X$ be a linear operator of approximation from above, i.e. $Qf\geqq f$ for any $f\in X$. Then Q is the identity in X. Indeed, assume that $Qf\neq f$ for some $f\in X$. From $Q(-f)\geqq -f$ we have $-Q(-f)\leqq f$ and hence $-Q(-f)\neq Q(f)$ which contradicts $0=Q(0)=Q(f-f)=Q(f)+Q(-f)$. In other words there are no sensible linear operators for onesided approximation.

5.4. With some additional efforts the present ideas can be modified for obtaining algebraic operators for onesided approximation. The necessary changes are: 1) $\tau_k(f;t)_p$ has to be replaced by a characteristic sensitive to the better abilities of the algebraic polynomials for approximation near to the boundary of the domain; 2) instead of Q_n from (3.8) to find an operator giving the order of the best algebraic approximation; 3) the supremums in (3.10) to be taken not on a uniform mesh but on parallelepipeds getting thinner near to the boundary; 4) $\Phi_n(x-2\pi j/n)$ in (3.10) to be replaced by a family of functions $\Phi_{j,n}(x)$ corresponding to the modifications in 3). Points 1), 3) and 4) from this program can be easily fulfilled. In 2) for k=1 Q_n can be replaced by a modification of Jackson operators. But for bigger k the form of the operators preserving P_{k-1} and giving the order of best algebraic approximation is rather complicated. This is the main difficulty to be overcomed in the program sketched above.

References

1. A.S.Andreev, V.A.Popov, Bl.Sendov. Jackson's type theorems for onesided polynomial and spline approximation. Comp. rend. Acad. bulg. Sci. 30,1977,11,1533-1536.
2. D.P.Dryanov. Direct and converse theorems for onesided approximation by entire functions. Comp. rend. Acad. bulg. Sci. 39, 1986,2,23-25.
3. G.Freud. Uber einseitige Approximation durch Polynome I. Acta Sci. Math. Szeged. 16,1955,12-18.
4. T.Ganelius. On onesided aproximation by trigonometrical polynomials. Math. Scand. 4,1956,247-258.
5. K.G.Ivanov. On the behaviour of two moduli of functions. II. Serdica Bulg. math. publ. 12, 1986, 196-203.
6. H.Johnen, K.Scherer. On the equivalence of the K-functional and moduli of continuity and some applications. Lecture Notes in Mathematics. Springer, Berlin-Heidelberg-New York. v.571, 1977, 119-140.
7. V.A.Popov. Converse theorem for the onesided trigonometrical approximations. Comp. rend. Acad. bulg. Sci. 30, 1977, 11, 1529-1532.
8. V.A.Popov. Onesided approximation of periodic functions of several variables. Comp. rend. Acad. bulg. Sci. 35, 1982, 12, 1639-1642.
9. V.A.Popov. Function spaces, generated by the averaged moduli of smoothness. PLISKA. 5, 1983, 132-143.
10. V.A.Popov, A.S.Andreev. Stečkin's tyre theorems for onesided trigonometrical and spline approximation. Comp. rend. Acad. bulg. Sci. 31, 1978, 2, 151-154.
11. О.В.Бесов,В.П.Ильин, С.М.Никольский. Интегральные представления функций и теоремы вложения. Наука, Москва, 1975.
12. Д.П.Дрянов.-Прави и обратни теореми и класове на насищане в теорията на апроксимациите. Дисертация, София, 1984.
13. А. Зигмунд. Тригонометрические ряды, т.1, 2, Наука, Москва,1965.
14. С.М.Никольский. Приближение функций многих переменных и теоремы вложения. Наука, Москва, 1977.
15. В.А.Попов. Односторонный К-функционал - многомерный случай. В сб. "Теория приближения функций", Труды международной конференций, Киев-1983, Наука, Москва, 1987.

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