CONSTRUCTIVE THEORY OF FUNCTIONS'87

SOFIA, 1988

SOME PROPERTIES OF GENERALIZED B-SPLINES

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1. Introduction. Multivariate B-splines were first introduced by de Boor (1976) extending a geometric interpretation of the classical univatriate B-spline, given by Curry and Schoenberg (1966). Properties of multivariate B-splines were further investigated mostly by Micchelli (1980), Dahmen (1980), Dahmen and Micchelli (1981), Höllig (1982), Hakopian (1982), C. de Boor and K. Höllig (1981). For an extensive survey on the subject we refer to Dahmen & Micchelli (1983).

The multivariate B-spline was recently given a probabilistic interpretation by the authors of this note. To elaborate on this we'll need the notion of Dirichlet distributed random variables. Recall that the random variables $\theta_0, \theta_1, \dots, \theta_n$ have the joint Dirichlet distribution $D(g_0, g_1, \dots, g_n)$ with parameters

(1.1) $g_0 > 0,\ g_1 > 0, \dots, g_n > 0,$

$((\theta_0, \theta_1, \dots, \theta_n) \in D(g_0, g_1, \dots, g_n))$ if $\theta_0 = 1 - \theta_1 - \dots - \theta_n$, and the joint probability density of $\theta_1, \theta_2, \dots, \theta_n$ with respect to the Lebesque measure on the simplex

$$S^n = \{(u_1, \dots, u_n) : u_i \geq 0,\ \sum_{i=1}^{n} u_i \leq 1\},\ u_0 = 1 - u_1 - \dots - u_n \text{ is}$$

$$\frac{\Gamma(g_0 + \dots + g_n)}{\Gamma(g_0) \dots \Gamma(g_n)} (1 - u_1 - u_2 - \dots - u_n)^{g_0 - 1} u_1^{g_1 - 1} \dots u_n^{g_n - 1}$$ ($\Gamma(.)$ is the well known Gamma function).

Let us introduce some useful notation. For a given set $A \subset R^s$, denote by $[A]$, $\dim(A)$ the closed convex hull and the dimension of A, respectively.

If now $t_0, \dots, t_n \in R^s$, $n \geq s$, are pairwise distinct and $\dim([\{t_0, \dots, t_n\}]) = s$ then the B-spline with knots $t_0, \dots, t_n$ having multiplicities, correspondingly $g_0, \dots, g_n$, i.e., $M(t; \underbrace{t_0, \dots, t_0}_{g_0}, \dots, \underbrace{t_n, \dots, t_n}_{g_n})$ coincides with the density $f_S(t)$ of the random vector $S = \theta_0 t_0 + \dots + \theta_n t_n$ with respect to the s-dimensional Lebesque measure, where $(\Theta_0,\Theta_1,\dots,\Theta_n)\in D(g_0,g_1,\dots,g_n)$ (cf Ignatov and Kaishev 1985 a,b).

Note, that probabilistically viewed, the quantities g_i, $i = 0,\dots,n$ may be arbitrary (cf.(1.1)). If we assume them to be integer, then $f_S(x)$ coincides with the classical B-spline. However, if g_i are arbitrary then $f_S(t)$ can be interpreted as a generalized B-spline, i.e., one allowing noninteger multiplicities g_i, $i = 0,\dots,n$ of the knots $t_0,\dots,t_n$.

Our aim in this paper is to illustrate how B-splines could be investigated using their relation to the linear transformation S of Dirichlet distributed random variables. In Section 2 we give a divided difference representation of the generalized B-spline in the univariate case, i.e., when $s = 1$ and discuss shortly some of its properties. The probabilistic interpretation of B-splines combined with a result of van Zwet (1979), concerning linear combinations of order statistics is used in Section 3 to derive an expansion for the integral of a univariate B-spline. In Section 4 an interesting formula which relates multivariate and univariate B-splines with especially chosen knots is derived. Further, a determinant formula for the multivariate B-spline is also given, combining its probabilistic interpretation with a result of Ali and Mead (1969) and Watson (1956).

2. Generalized B-splines. Recall, that Curry and Schoenberg (1966) defined the univariate B-spline $M(t;\underbrace{t_0,\dots,t_0}_{g_0},\dots,\underbrace{t_n,\dots,t_n}_{g_n})$ with knots $t_0,\dots,t_n\in R^1$ having positive, integer multiplicities, correspondingly, $g_0,\dots,g_n$, as the 1-1-th divided difference of the function $\Phi(u) = (1-1)(u-t)_+^{1-2}$, i.e.,

$$(2.1)\quad M(t;\underbrace{t_0,\dots,t_0}_{g_0},\dots,\underbrace{t_n,\dots,t_n}_{g_n}):=[\underbrace{t_0,\dots,t_0}_{g_0},\dots,\underbrace{t_n,\dots,t_n}_{g_n}]_u\Phi(u)$$

where $(z)_+ = \max\{0,z\}$, $1 = g_0+\dots+g_n$.

Here we give a divided difference expression for the univariate, i.e., when $s = 1$, generalized B-spline. For the purpose denote by $\hat{g}_i$ the integer part of g_i, $\bar{g}_i = g_i - \hat{g}_i$. Suppose, g_0, $g_1,\dots,g_n$, $g_{n+1},\dots,g_{n+m}$ are such that $\bar{g}_i > 0$, $i = 0,1,\dots,n$; $\bar{g}_i = 0$, $i = n+1,\dots,n+m$. Denote by $1 = \sum_{i=0}^{n+m}\hat{g}_i$, $(1 > 1)$.

The generalized B-spline

(2.2) $$M(t;\underbrace{t_0,\dots,t_0}_{g_0},\dots,\underbrace{t_{n+m},\dots,t_{n+m}}_{g_{n+m}})$$

$$= [\underbrace{t_0,\dots,t_0}_{\hat{g}_0},\dots,\underbrace{t_n,\dots,t_n}_{\hat{g}_n},\underbrace{t_{n+1},\dots,t_{n+1}}_{g_{n+1}},\dots,\underbrace{t_{n+m},\dots,t_{n+m}}_{g_{n+m}}]\,u^{H(u)},$$

where $t \in [B]$, B is the set of all t_i-s for which $\hat{g}_i > 1$, $H(u) =$

$$\frac{\Gamma(g_0+\dots+g_{n+m})}{\Gamma(1-1)\Gamma(\bar{g}_0)\dots(\bar{g}_n)}\int\dots\int_{\Delta_n}(u-t+\sum_{i=0}^{n}(t_i-u)y_i)_+^{1-2}\,y_0^{\bar{g}_0-1}\dots y_n^{\bar{g}_n-1}\,dy_0\dots dy_n,$$

and

$\Delta_n = \{(y_0,\dots,y_n : 0 < y_i,\ i = 1,\dots,n,\ y_0+\dots+y_n < 1\}$.

For a proof of this divided difference representation we refer to a forthcoming paper by Ignatov and Kaishev (1987). Note, that if $\bar{g}_i = 0$, $i = 0,\dots,n+m$, then it is directly verified that (2.2) coincides with the expression (2.1) for the classical B-spline. However, if $\bar{g} > 0$ then it can be checked that the divided difference in (2.2) does not vanish for every $t \notin [B]$, unlike (2.1), which is zero everywhere on R^1 except for the interval $[t_0, t_n]$.

According to the probabilistic interpretation of B-splines, given in Section 1 the generalized B-spline in (2.2) coincides with the density $f_S(t)$ of the random variable $S = \sum_{i=0}^{n+m} t_i\theta_i$, where $(\theta_0,\theta_1,\dots,\theta_{n+m}) \in D(g_0,g_1,\dots,g_{n+m})$ and $t_i \in R^1$, $i = 0,\dots,n+m$.

It is worth mentioning that the generalized B-spline (2.2) admits a representation as a divided difference of an elementary function only if one of the knots is of noninteger multiplicity, i.e., when n = 0. In this case, if $t_0 < t_i$, $i = 1,\dots,m$, we have (cf Ignatov & Kaishev 1986)

$$M(t;\underbrace{t_0,\dots,t_0}_{g_0},\dots,\underbrace{t_m,\dots,t_m}_{g_m}) =$$

$$= [\underbrace{t_0,\dots,t_0}_{\hat{g}_0},\underbrace{t_1,\dots,t_1}_{g_1},\dots,\underbrace{t_m,\dots,t_m}_{g_m}]\,u^{H'(u)},$$

where $H'(u) = (1 + \bar{g}_0 - 1)(u - t_0)^{-\bar{g}_0}(u - x)_+^{1+\bar{g}_0-2}$.

A numerically useful expression for generalized B-splines with only two knots of noninteger multiplicity, i.e., when n = 1, is to be found in Ignatov and Kaishev (1987).

3. <u>An expansion for B-slines</u>. We shall now use two theorems of asymptotic statistics to establish easily results for B-splines.

Let $a_1(n),\dots,a_n(n)$ be a real sequence with $x_{0,n} = 0$, $x_{1,n} = a_1(n)$

$x_{2,n} = a_1(n) + a_2(n),\ldots,x_{n,n} = a_1(n)+\ldots+a_n(n)$ and $\sum_{j=1}^{n} a_j^2(n) \neq 0$ for every n. Define $\bar{x}_n = (n+1)^{-1}\sum_{j=0}^{n} x_{j,n}$, $\sigma_n^2 = \sum_{j=0}^{n}(x_{j,n} - \bar{x}_n)^2/(n+1)(n+2)$.

$t_{j,n} = (x_{j,n} - \bar{x}_n)/\sigma_n$, $j = 0,\ldots,n$.

Since $\Sigma\, a_j^2(n) \neq 0$ implies that $\sigma_n^2 \neq 0$ the $t_{j,n}$'s are well defined with $\Sigma\, t_{j,n} = 0$ and $\Sigma\, t_{j,n}^2 = (n+1)(n+2)$.

THEOREM 3.1.

$$\lim_{n\to\infty} \sup_{-\infty<x<\infty} \left| M(x;t_{0,n},t_{1,n},\ldots,t_{n,n}) - (2\pi)^{-1/2}\exp(-x^2/2)\right| = 0$$

if and only if

$$\lim_{n\to\infty} \left(\max_{0\leq j\leq n} \left| t_{j,n}((n+1)(n+2))^{-1/2}\right|\right) = 0. \tag{3.1}$$

PROOF.

Applying the probabilistic interpretation of the B-spline, given in Section 1 to a theorem of Hecker (1976) which gives a necessary and sufficient condition for the asymptotic normality of the linear combination $S_n = t_{0,n}\,\theta_0+\ldots+t_{n,n}\,\theta_n$, $(\theta_0,\ldots,\theta_n)\in D\underbrace{(1,\ldots,1)}_{n+1}$ we easily obtain the assertion of Theorem 3.1.

Under the condition (3.1) we shall derive an expansion for the spline distribution function

$$\int_{-\infty}^{x} M(t;t_{0,n},t_{1,n},\ldots,t_{n,n})dt$$

using a theorem of van Zwet (1979).

Define

$$\Phi(x) = \int_{-\infty}^{x} \varphi(t)dt,$$

$$\varphi(x) = (2\pi)^{-1/2}\exp(-x^2/2),$$

H_r - the Hermite polynomial of degree r, i.e., $d^r\varphi(x)/dx^r = (-1)^r\varphi(x)H_r(x)$. For $j = 0,\ldots,n$ and real x let

$$V_{j,n}(x) = x_{j,n} - \bar{x}_n - \sigma_n x,$$

$$W_{j,n}(x) = V_{j,n}(x)\{\sum_{j=0}^{n} V_{j,n}^2(x)\}^{-1/2}$$

$$= ((n+2)/(n+2+x^2))^{1/2}(t_{j,n} - x)((n+1)(n+2))^{-1/2},$$

$$\xi(x) = -\sum_{j=0}^{n} W_{j,n}(x) = x((n+1)/(n+2+x^2))^{1/2}.$$

It is easy to check that $\Sigma W_{j,n}^2(x) = 1$. For integer $m \geq 3$ and real x and z, let

$$G_{m,n}(z,x) = \Phi(x) - \varphi(z)\Sigma^* H_{(\Sigma k\nu_k - 1)}(z) \prod_{k=3}^{m-1} (1/\nu_k!)(k^{-1} \sum_{j=0}^{n} W_{j,n}^k(x))^{\nu_k},$$

where Σ^* denotes summation over all nonnegative integers $\nu_3,\ldots,\nu_{m-1}$ with

$$1 \leq \sum_{k=3}^{m-1} (k-2)\nu_k \leq m-3.$$

THEOREM 3.2.

For every integer $m \geq 3$ there exists a constant C_m such that for every $n = 1,2,\ldots$ and every $a_1(n),\ldots,a_n(n)$ with $\Sigma a_j^2(n) \neq 0$,

$$\sup_x \left| \int_{-\infty}^{x} M(t;t_{0,n},t_{1,n},\ldots,t_{n,n})dt - G_{m,n}(\xi(x),x)\right| \leq C_m \sum_{j=0}^{n} \left|t_{j,n}((n+1)(n+2))^{-1/2}\right|^m.$$

PROOF.

Follows by the probabilistic interpretation of the B-splines (see Section 1) and the theorme of van Zwet (1979) establishing the Edgeworth expansion for the distribution function of S_n.

Since we require the condition (3.1) to be fulfilled the remainder in the above expansion tends to zero for every $m \geq 3$ and for increasing m. The rates will depend of course on the triangular array $\{a_j(n)\}$ related to the knots of the B-spline. The remainder approximately behaves as $n^{-1/2(n-2)}$ (see van Zwet (1979)).

Following van Zwet (1979) we develop the expansion of Theorem 3.2 for the special cases $m = 3$ and $m = 5$. For $m = 3$ it reduces to a bound of Berry-Essenn type, typically of order $n^{-1/2}$.

COROLLARY 1.

There exists a constant C such that for every $n = 1,2,\ldots$ and every $a_1(n),\ldots,a_n(n)$ with $\sum_{j=1}^{n} a_j^2(n) \neq 0$,

$$\sup_x \left| \int_{-\infty}^{x} M(t;t_{0,n},t_{1,n},\ldots,t_{n,n})dt - \Phi(x)\right| \leq C \sum_{j=0}^{n} \left|t_{j,n}((n+1)(n+2))^{-1/2}\right|^3.$$

Let now $m = 5$. Define

$$F_n(x) = \Phi(x) - \varphi(x)((1/3)\sum_{j=0}^{n}(t_{j,n}^3((n+1)(n+2))^{-1/2})^3 \cdot H_2(x) +$$

$$+ ((1/4)\sum_{j=0}^{n}(t_{j,n}((n+1)(n+2))^{-1/2})^4 - \frac{1}{2n})H_3(x) +$$

$$+ (1/18)(\sum_{j=0}^{n}(t_{j,n}((n+1)(n+2))^{-1/2})^3)^2 H_5(x)).$$

COROLLARY 2.

There exists a constant C such that for every n = 1,2,... and every $a_1(n),\dots,a_n(n)$ with $\sum_{j=1}^{n} a_j^2(n) \neq 0$,

$$\sup_x \left|\int_{-\infty}^{\infty} M(t;t_{0,n},t_{1,n},\dots t_{n,n})dt -\right.$$

$$\left. - F_n(x)\right| \leq C \sum_{j=0}^{n} |t_{j,n}((n+1)(n+2))^{-1/2}|^5.$$

Corollaries 1 and 2 follow from the probabilistic interpretation of the B-spline and Corollaries 1 and 2 of van Zwet (1979).

The remainder term in the last expansion is given by van Zwet (1979) to be typically of order $n^{-3/2}$.

We finally remark that the Berry-Essenn bound of Corollary 1 can be sharpened by means of Corollary 2 to obtain

$$\sup_x \left|\int_{-\infty}^{\infty} M(t;t_{0,n},t_{1,n},\dots,t_{n,n})dt -\right.$$

$$\left. - \Phi(x)\right| \leq C\ (\left|\sum_{j=0}^{n}(t_{j,n}((n+1)(n+2))^{-1/2})^3\right| +$$

$$+ \sum_{j=0}^{n}(t_{j,n}((n+1)(n+2))^{-1/2})^4).$$

4. New representations for multivariate B-splines. The distribution of the random vector S of (1.2) has been investigated by several authours. Formulae for the density $f_S(t)$ were obtained by Quenouille (1949), G. Watson (1956), Ali and Mead (1969). Since $f_S(t)$ coincides with the multivariate B-spline, the latter could be directly expressed by the formulae for $f_S(t)$.

Here we present such an expression using, for instance, the formula of Ali and Mead (1969). Let the knots $t_0,\dots,t_n \in R^s$, $(s \geq 1)$ and let also $t_0 = (\underbrace{0,\dots,0}_{s})'$.

The multivariate B-spline is expressed in terms of certain determinants as

$$M(x;t_0,\dots,t_n) = \frac{n!}{(n-s)!} \sum_{i_1=0}^{n} \sum_{\substack{i_2=0\\ i_2\neq i_1}}^{n} \dots \sum_{\substack{i_s=0\\ i_s\neq i_j,\\ j=1,\dots,s-1}}^{n} \left(\prod_{k=1}^{s} \nu_k\right).$$

$$\frac{\begin{vmatrix} 1 & t_{1,i_1} & \dots & t_{s,i_1} \\ \dots & \dots & \dots & \dots \\ 1 & t_{1,i_s} & \dots & t_{s,i_s} \\ 1 & x_1 & \dots & x_s \end{vmatrix}^{n-s}}{\prod\limits_{\substack{i_{s+1}=0\\ i_{s+1}\neq i_j\\ j=1,\dots,s}}^{n} \begin{vmatrix} 1 & t_{1,i_1} & \dots & t_{s,i_1} \\ \dots & \dots & \dots & \dots \\ 1 & t_{1,i_{s+1}} & \dots & t_{s,i_{s+1}} \end{vmatrix}},$$

provided $\begin{vmatrix} 1 & t_{1,i_1} & & t_{j,i_1} \\ \dots & \dots & \dots & \dots \\ 1 & t_{1,i_{j+1}} & \dots & t_{j,i_{j+1}} \end{vmatrix} \neq 0$ for

$j = 1,\dots,s$ for all distinct $i_1,\dots,i_{j+1}$, where $\nu_k = \nu(X_k)$,

$$k=1,\dots,s;\quad \nu(y) = \begin{cases} 1 & y \geq 0 \\ 0 & y < 0, \end{cases}$$

$$X_k = \frac{\begin{vmatrix} 1 & t_{1,i_1} & \dots & t_{k,i_1} \\ \dots & \dots & \dots & \dots \\ 1 & t_{1,i_k} & \dots & t_{k,i_k} \\ 1 & x_1 & \dots & x_k \end{vmatrix}}{\begin{vmatrix} 1 & t_{1,i_1} & \dots & t_{k-1,i_1} \\ \dots & \dots & \dots & \dots \\ 1 & t_{1,i_k} & \dots & t_{k-1,i_k} \end{vmatrix}}$$

$x = (x_1,\dots,x_s)'$, $t_i = (t_{1,i},\dots,t_{s,i})'$, $i = 0,\dots,n$.

Finally we give an interesting formula, relating multivariate B-spline to univariate one for a special choice of the corresponding knot sets. To this end, let the vectors $t_i = (t_{i,1},\dots,t_{i,s})'$, $i = 0,\dots,s-1$ have real entries and denote $T = \|t_{i,j}\|$, $T^{-1} = \|t^*_{i,j}\|$, $i = 0,\dots,s-1$ $j = 1,\dots,s$. Denote also $\overline{0} = \underbrace{(0,\dots,0)'}_{s}$.

The multivariate B-spline with knots $t_0,\dots,t_{s-1},t_s,\dots,t_{n-1} \in R^s$, $t_i \equiv \overline{0}$, $i \equiv s,\dots,n-1$ at the point $x = (x_1,\dots,x_s)' = Ta$, $a = (a_1,\dots,a_s)'$, $0 < a_i$ is given as

$$M(x;t_0,\dots,t_{s-1},t_s,\dots,t_{n-1}) = \frac{(n-1)!}{|\det T|} M(y;y_0,\dots,y_{n-s}),$$

where $y = 1 - \sum_{i=1}^{s} x_i \sum_{j=0}^{s-1} t^*_{j,i} = 1 - \sum_{i=1}^{s} a_i$, and $y_i = i$, $i = 0,\dots,n-s$.

The proof of this formula will appear elsewhere.

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