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APPROXIMANTS FOR FUNCTIONS REPRESENTED BY LIMIT PERIODIC CONTINUED FRACTIONS

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1. Introduction. Let us start with a little example. Let $f(z)$ be a meromorphic function with branch points at $c \in \mathbb{C}$ and ∞ of order 1. Let further $r(z)$ be a rational approximant to $f(z)$. Even though $r(z)$ may approximate $f(z)$ very well in part of the domain of f, it can only approximate one meromorphic branch of f, since it has no branch points. We must also accept the approximation to be poorer in parts of a neighborhood of some branch cut for $f(z)$.

Let $\mathcal{R}(\sqrt{c-z})$ denote the field of functions rational in z and $\sqrt{c-z}$. By picking approximants from this extended field, we might do better. But we need a "method to pick".

Assume next that $K(a_n/b_n)$ is a continued fraction expansion of f with polynomial elements a_n, b_n. The approximants

$$(1.1)\qquad f_n = \frac{a_1}{b_1} + \frac{a_2}{b_2} + \dots + \frac{a_n}{b_n} = \frac{A_n}{B_n}$$

of $K(a_n/b_n)$ are rational functions. Indeed, A_n and B_n are polynomials satisfying the recurrence relation

$$(1.2)\qquad \begin{pmatrix} A_n \\ B_n \end{pmatrix} = b_n \begin{pmatrix} A_{n-1} \\ B_{n-1} \end{pmatrix} + a_n \begin{pmatrix} A_{n-2} \\ B_{n-2} \end{pmatrix} \text{ with } \begin{pmatrix} A_{-1} & A_o \\ B_{-1} & B_o \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Assume that $f_n(z) \to f(z)$ in some domain D. Then f_n are rational approximations to f in D. Clearly, the nth tail

$$(1.3)\qquad \frac{a_{n+1}}{b_{n+1}} + \frac{a_{n+2}}{b_{n+2}} + \frac{a_{n+3}}{b_{n+3}} + \dots .$$

also converges in D. Let $f^{(n)}$ denote its value. Then

$$(1.4)\qquad f = \frac{a_1}{b_1} + \frac{a_2}{b_2} + \dots + \frac{a_n}{b_n + f^{(n)}} = \frac{A_n + A_{n-1} f^{(n)}}{B_n + B_{n-1} f^{(n)}} = S_n(f^{(n)}).$$

This relation obviously holds for all $z \in D$. However, since $f^{(n)}(z)$ is the image of $f(z)$ under the linear fractional transformation S_n^{-1}, it has a natural extension to the domain of f, and (1.4) still holds. Clearly $f^{(n)}$ also has branch points at c and ∞ of order 1.

Constructing f_n is to chop off the nth tail (with its branch points). By replacing $f^{(n)}$ by some element $w_n \in R(\sqrt{c-z})$ "resembling" $f^{(n)}$, we get a modified approximant

$$(1.5) \qquad S_n(w_n) = \frac{a_1}{b_1} + \frac{a_2}{b_2} + \ldots + \frac{a_n}{b_n+w_n} = \frac{A_n+A_{n-1}w_n}{B_n+B_{n-1}w_n} \in R(\sqrt{c-z}).$$

For several cases there exist methods for picking such w_ns.

In this paper we shall see what can be gained by such an extension of the approximant field. To keep the arguments simple, we shall consider the case where a function f has a continued fraction expansion of the form

$$(1.6) \qquad 1 + K(a_n z/1) \text{ where } \alpha_n \in \mathbb{C}\setminus\{0\},\ \alpha_n \to \alpha \in \mathbb{C}\setminus\{0\}.$$

Examples of such functions are ${}_2F_1(a, 1; c; z)$ and ${}_2F_1(a, b; c; z)/{}_2F_1(a, b+1; c+1; z)$ where $a, b, c, c-a, c-b \in \mathbb{C}\setminus\mathbb{Z}_0^-$, [7, Theorem 6.1 p.199]. The ideas may however be carried out in more general situations. (See for instance [1, 2].) Please note also that if $\{r_n\}$ is a sequence of rational approximations of increasing order, then one can easily write up continued fractions whose sequence of approximants is exactly $\{r_n\}$. If one of these continued fractions has a form we can handle, we are in business.

In Section 2 we review some results on limit periodic continued fractions with constant elements. In Section 3 we apply these to (1.6). The proof of the results in Section 3 are found in Section 4.

__2. Limit periodic continued fractions.__ Let $K(a_n/1)$ be a continued fraction with $a_n \in \mathbb{C}\setminus\{0\}$ such that $a_n \to a \in E = \{z \in \mathbb{C}\setminus\{0\};\ |\arg(z+1/4)| < \pi\}$.

A. The periodic continued fraction

$$(2.1) \qquad K\frac{a}{1} = \frac{a}{1} + \frac{a}{1} + \frac{a}{1} + \ldots$$

is easily proved to converge to the value

$$(2.2) \qquad \Gamma = (\sqrt{1+4a} - 1)/2, \text{ where } \operatorname{Re}\sqrt{\ } > 0,$$ [7, Theorem 3.2, p.48].

(Γ is the attractive fixed point of the linear fractional transformation $s(w) = a/(1+w)$.) $K(a_n/1)$ "resembles $K(a/1)$ sufficiently" to inherit the property of convergence. That is, $K(a_n/1)$ converges to a value $f \in \hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, [10, Satz 2.41, p.93].

B. From A. follows also that all the tails of $K(a_n/1)$ converge. Let $f^{(n)}$ denote the value of its nth tail. Then it is natural to think that for large n, $f^{(n)}$ will be close to Γ. And indeed, we have that $f^{(n)} \to \Gamma$, [10,Satz 2.41, p.93].

C. The values $f^{(n)}$ in B. satisfy the recurrence relation

$$g^{(n-1)} = \frac{a_n}{1+g^{(n)}} \quad \text{for } n=1,2,3,... \tag{2.3}$$

with $g^{(o)} = f = f^{(o)}$. Any other sequence $\{g^{(n)}\}$ satisfying (2.3) is called a sequence of wrong tails for $K(a_n/1)$. For these we know that $g^{(n)} \to -(1+\Gamma)$, i. e. the repulsive fixed point of $s(w)$. (This follows since $-(1+\Gamma)$ is the attractive fixed point of $s^{-1}(w)$.) [3, Theorem 3.1B(ii), p.565].

D. In view of (1.4) and the fact that $f^{(n)} \to \Gamma$ we expect $S_n(\Gamma)$ to approximate f better than $f_n = S_n(0)$. And so it does, at least from some n on, since

$$\frac{f-S_n(\Gamma)}{f-f_n} \to 0 \text{ if } f \neq \infty, \tag{2.4}$$

[11, Theorem 2.1 + an extra argument].

E. Let $\varepsilon_n = f^{(n)} - \Gamma$. Since $f = S_n(f^{(n)})$, it seems natural, as a next step, to try to estimate ε_n to obtain good approximants $S_n(w_n)$ where $w_n = \Gamma + \hat{\varepsilon}_n$. To do so, we need to use information on how $\delta_n = a_n - a$ approaches 0. It is e. g. known that if $\delta_{n+1}/\delta_n \to t \in \mathbb{C}$, then $\delta_{n+1}/\varepsilon_n \to 1 + \Gamma + \Gamma t$, [6, 8]. Hence, $\hat{\varepsilon}_n = \delta_{n+1}/(1 + \Gamma + t\Gamma)$ seems to be a useful estimate under this condition. And indeed, it turns out that

$$\frac{f-S_n(\Gamma+\hat{\varepsilon}_n)}{f-S_n(\Gamma)} \to 0 \text{ if } f \neq \infty, \tag{2.5}$$

[5, 8].

These results can all be stated in more general forms. First of all, they represent special cases of what happens for limit k-periodic continued fractions. Moreover, A., B., C. and D. can be regarded as consequences of the continuity of $f = K(a_n/b_n)$ regarded as a function of its elements $a_1, b_1, a_2, b_2, \ldots$ (See [4]).

3. Limit periodic C-fractions. From A. in Section 2 follows directly that the limit periodic C-fraction $K(\alpha_n z/1)$ given by (1.6) converges pointwise to a function $f(z)$ in the domain $D_\alpha = \{z \in \mathbb{C} \setminus \{0\}; |\arg(\alpha z+1/4)| < \pi\}$. By use of the uniform parabola theorem [7, Theorem 4.40, p.99] follows even that the convergence is locally uniform in $\tilde{D}_\alpha = \{z \in D_\alpha; f(z) \neq \infty\}$, that f and $f^{(n)}$ are meromorphic functions in D_α, and that their singularities at 0 are removeable.

What happens if we, instead of the rational approximants $f_n = S_n(0)$ of $K(\alpha_n z/1)$, start using the modified approximants $S_n(\Gamma) \in R(\sqrt{1+4\alpha z})$, where $\Gamma(z) = (\sqrt{1+4\alpha z} - 1)/2$?

First of all, we notice by (2.4) that $S_n(\Gamma)$ converges faster to f than f_n for all $z \in \tilde{D}_\alpha$. Upper bounds for the ratio $|f-S_n(\Gamma)|/|f-f_n|$ (under additional conditions) can be found in [11,Theorem 2.1]. The following result gives an estimate for the asymptotic improvement obtained by changing over from f_n to $S_n(\Gamma)$:

THEOREM 3.1.

$$\left|\frac{f-S_n(\Gamma)}{f-f_n}\right| < Q_n \sim \frac{|1+\Gamma|/|\Gamma|}{(|1+\Gamma|-|\Gamma|)|1+2\Gamma|}\,|z|\,\max_{m>n}|\alpha_m-\alpha| \text{ as } n \to \infty \tag{3.1}$$

for all $z \in \tilde{D}_\alpha$.

As is directly seen from (3.1), there are cases where the improvement is not so impressive, but please observe the following four points:

(i) $S_n(\Gamma)$ can be computed instead of $f_n = S_n(0)$ by the same algorithms (involving the same number of operations) as long as Γ is computed.

(ii) $S_n(\Gamma)$ is still a reasonably simple approximant.

(iii) The improvement works in the whole domain $\tilde{D}_\alpha$

(iv) (3.1) can be interpreted in the following way. $K(\alpha z/1)$ is an auxiliary continued fraction whose tail values $\tilde{f}^{(n)} = \Gamma$ are known. The closer $K(\alpha_n z/1)$ is to our auxiliary continued fraction, the better its modified approximants $S_n(\tilde{f}^{(n)})$ behave. Thus we could improve $S_n(\Gamma)$ if we found a better auxiliary continued fraction. We shall come back to this point later.

The next question is: Will $S_n(\Gamma)$ converge also beyond D_α? Or, to put it a little stronger, will the new approximants $S_n(\Gamma)$ app-

roximate a possible analytic (or meromorphic) continuation of f? It turns out that the nearness of the two continued fractions $K(\alpha_n z/1)$ and $K(\alpha z/1)$ to each other is of vital importance. From [12, Theorems 4.1 and 4.2]
follows that if

$$(3.2) \qquad |\alpha_n - \alpha| < \gamma r^n \text{ for all } n, \text{ where } \gamma > 0 \text{ and } 0 < r < 1,$$

then the following hold:

A. $f(z)$ has a meromorphic continuation $f^*(z)$ to the Riemann surface $D^*_{\alpha,r} = \{z;\ |1-(1+4\alpha z)^{\frac{1}{2}}|/|1+(1+4\alpha z)^{\frac{1}{2}}| < 1/r\}$.

B. f^* has a branch point of order 1 at $z = -1/4\alpha$. Indeed, $f(z) = F(1+4\alpha z)^{\frac{1}{2}})$ where $F(w)$ is meromorphic in $E^*_{\alpha,r} = \{w \in \mathbb{C};\ |1-w|/|1+w| < 1/r\}$.

C. $S^*_n(\Gamma^*) \to f^*$ for all $z \in D^*_{\alpha,r}$, where $S^*_n(\Gamma^*)$ is the analytic continuation of $S_n(\Gamma)$ to $D^*_{\alpha,r}$.

This seems to indicate that in order to approximate $f(z)$ beyond D_α, we need to introduce a branch point at $z = -1/4\alpha$ of the right order in the approximants. Thinking in terms of auxiliary continued fractions again, we can interprete this result as follows. The auxiliary continued fraction $K(\alpha z/1)$ either diverges or converges to $(\sqrt{1+4\alpha z} - 1)/2$ for $z \in D^*_{\alpha,r} \setminus D_\alpha$, whereas $\Gamma^*(z) = (-\sqrt{1+4\alpha z} - 1)/2$ for $z \in D^*_{\alpha,r} \setminus \overline{D_\alpha}$. However, its modified approximants

$$(3.3) \qquad \tilde{S}^*_n(\Gamma^*) = \frac{\alpha z}{1} + \frac{\alpha z}{1} + \dots + \frac{\alpha z}{1+\Gamma^*(z)} = \Gamma^*(z) \to \Gamma^*(z)$$

for all z. The closer $K(\alpha_n z/1)$ is to $K(\alpha z/1)$, the better $S^*_n(\Gamma^*)$ will imitate $\tilde{S}^*_n(\Gamma^*)$.

In Section 2.E. was shown an example of how one can find better modified approximants in more special situations. Or in other words, how to find a better auxiliary continued fraction $K(\hat{\alpha}_n(z)/1)$; $\hat{\alpha}_n = (\Gamma + \hat{\varepsilon}_{n-1})(1 + \Gamma + \hat{\varepsilon}_n)$ for $K(\alpha_n z/1)$; $\alpha_n z = (\Gamma + \varepsilon_{n-1})(1 + \Gamma + \varepsilon_n)$. Translated to our situation it means that if $\mu_{n+1}/\mu_n \to t$ where $\mu_n = \alpha_n - \alpha$, then

$$(3.4) \qquad S_n(\Gamma + \hat{\varepsilon}_n) \in R(\sqrt{1+4\alpha z}) \text{ where } \hat{\varepsilon}_n(z) = \frac{\mu_{n+1} z}{1+\Gamma(z)+t\Gamma(z)}$$

converges faster to $f(z)$ than $S_n(\Gamma)$ (which converges faster than $f_n = S_n(0)$) for all $z \in \tilde{D}_\alpha$. The following result says something about how much faster:

THEOREM 3.2.

If $\mu_{n+1}/\mu_n \to t$, then

$$(3.5) \qquad \left| \frac{f-S_n(\Gamma+\hat{\varepsilon}_n)}{f-S_n(\Gamma)} \right| < Q_n \sim \frac{(|\alpha z|+|\Gamma t|^2)}{|1+\Gamma|-|\Gamma t|} \max_{m\geq n} \left|t - \frac{\mu_{m+1}}{\mu_m}\right| +$$

$$+ \frac{2|\Gamma t z|}{(|1+\Gamma|-|\Gamma|)(|1+\Gamma|-|\Gamma t|)} \max_{m\geq n} |\alpha_m - \alpha|; \; z \in \tilde{D}_\alpha.$$

In favourable cases we can also obtain that meromorphic continuations of $S_n(\Gamma + \hat{\varepsilon}_n)$ converge to meromorphic continuations f^* of f, also at points where $S_n^*(\Gamma^*)$ fails to approximate f^*.

THEOREM 3.3.

If $\mu_{n+1}/\mu_n \to t$ with $|t| < 1$ and $|t - \mu_{n+1}/\mu_n| \leq cs^n$ for some $c>0$ and $0<s<1$, then the following hold:

A. $f(z)$ has a meromorphic continuation $f^*(z)$ to the Riemann surface $D^*_{\alpha,r} = \{z; |1 - (1+4\alpha z)^{\frac{1}{2}}|/|1 + (1+4\alpha z)^{\frac{1}{2}}| < 1/r\}$ where $r = |t| \cdot \max\{|t|, s\}$.

B. f^* has a branch point of order 1 at $z = -1/4\alpha$.

C. $S_n^*(\Gamma^* + \hat{\varepsilon}_n^*) \to f^*$ for all $z \in D^*_{\alpha,r}$, where $\Gamma^*, \hat{\varepsilon}_n^*$ and S_n^* denote the meromorphic continuations of Γ, $\hat{\varepsilon}_n$ and S_n respectively to $D^*_{\alpha,r}$.

REMARK. If $s < |t|$, then we obtain that $S_n^*(\Gamma^* + \hat{\varepsilon}_n^*)$ converges in $D^*_{\alpha,|t|^2}$, whereas $S_n^*(\Gamma^*)$ converges in $D^*_{\alpha,|t|}$.

There is a slight numerical problem for $z \in D^*_{\alpha,r} \setminus D_\alpha$ since the computation of $S_n^*(\Gamma^*)$ by (3.3) is unstable for these values of z. This is however often easy to solve. Just write up a continued fraction whose approximants are exactly $\{S_n^*(\Gamma^*)\}$, and compute its ordinary approximants by any of the existing algorithms.

4. Proofs.

To prove some of these results we shall use the following lemma:

LEMMA 4.1.

Let $K(a_n/b_n)$ be limit periodic such that

$$(4.1) \qquad a_n \to a \in \mathbb{C}\setminus\{0\}, \; b_n \to b \in \mathbb{C}\setminus\{0\}; \quad |\arg(1+4a/b^2)| < \pi,$$

and let $d_n = \max_{m\geq n} |a_m - a|$, $e_n = \max_{m\geq n} |b_m - b|$ and

$$(4.2) \qquad \Gamma = (\sqrt{1+4a/b^2} - 1)b/2 \; ; \quad \mathrm{Re}\sqrt{} > 0.$$

Let $e_N < D$ and $d_N + |\Gamma|e_N < (D - e_N)^2/4$ for some $N \in \mathbb{N} \cup \{0\}$, where $D = |b + \Gamma| - |\Gamma|$. Then

$$(4.3) \qquad |f^{(n)} - \Gamma| < \frac{2(d_n+|\Gamma|e_n)}{D-e_n+\sqrt{(D-e_n)^2-4(d_n+|\Gamma|e_n)}} \quad \text{for some } n \geq N.$$

PROOF: Let $V_n = \{z \in \mathbb{C};\ |z - \Gamma| \leq R_n\}$ where R_n is exactly the bound given by (4.3); $n \geq N$. Then $a_m/(b_m + V_n) \subseteq V_n$ for all $m \geq n$, since for $w \in V_n$ we have

$$(4.4) \qquad \left| \frac{a_m}{b_m+w} - \Gamma \right| = \left| \frac{a_m-a-\Gamma(b_m-b)-\Gamma(w-\Gamma)}{b+\Gamma+(b_m-b)+(w-\Gamma)} \right|$$
$$\leq \frac{d_n+|\Gamma|e_n+|\Gamma|R_n}{|b+\Gamma|-(e_n+R_n)} = R_n.$$

It is well known that $f^{(m)} \to \Gamma$. Hence $f^{(m)} \in V_n$ for m sufficiently large, and thus

$$(4.5) \qquad f^{(n)} = \frac{a_{n+1}}{b_{n+1}} + \frac{a_{n+2}}{b_{n+2}} + \dots + \frac{a_m}{b_m+f^{(m)}} \quad \in V_n$$

by use of the inclusion property $a_m/(b_m + V_n) \subseteq V_n$. This proves (4.3).

<u>PROOF of Theorem 3.1.</u> We shall use the standard formula

$$(4.6) \qquad \frac{f-S_n(x)}{f-S_n(y)} = \frac{h_n+y}{h_n+x} \cdot \frac{f^{(n)}-x}{f^{(n)}-y} \quad \text{if } f \neq \infty \text{ and } x \neq y,$$

where

$$(4.7) \qquad S_n(w) = \frac{a_1}{b_1} + \frac{a_2}{b_2} + \dots + \frac{a_n}{b_n+w} \quad , \quad h_n = -S_n^{-1}(\infty).$$

In our situation $\{-h_n(z)\}_{n=o}^{\infty}$ is a sequence of wrong tails for $K(\alpha_n z/1)$ for all $z \in \tilde{D}_\alpha$, and thus, by part C. in Section 2, $h_n(z) \to 1 + \Gamma(z)$ in $\tilde{D}_\alpha$. Since $f^{(n)}(z) \to \Gamma(z)$ in $\tilde{D}_\alpha$, and $|f^{(n)}-\Gamma| \leq R_n \sim d_n/D$ by (4.3), (3.1) follows.

<u>PROOF of Theorem 3.2.</u> Since $\mu_{n+1}/\mu_n \to t$ it follows that $\mu_{n+1} \neq 0$ and thus $\Gamma(z) \neq \Gamma(z) + \hat{\varepsilon}_n(z)$ for all $z \in \tilde{D}_\alpha$ for sufficiently large n. From (4.6) follows therefore that

$$(4.8) \qquad \frac{f-S_n(\Gamma+\hat{\varepsilon}_n)}{f-S_n(\Gamma)} \sim \frac{f^{(n)}-\Gamma-\hat{\varepsilon}_n}{f^{(n)}-\Gamma} = 1 - \frac{\mu_{n+1}z/\varepsilon_n}{1+\Gamma+t\Gamma} .$$

Since $\alpha_{n+1}z = f^{(n)}(1 + f^{(n+1)})$ and $\alpha z = \Gamma(1+\Gamma)$, it follows that

$$(4.9) \qquad \mu_{n+1}z = (1+\Gamma)\varepsilon_n + \Gamma\varepsilon_{n+1} + \varepsilon_{n+1}\varepsilon_n,$$

and hence, $\mu_{n+1}z/\varepsilon_n = 1+\Gamma+\varepsilon_{n+1}+\Gamma\varepsilon_{n+1}/\varepsilon_n$. Inserted into (4.8), we get that

$$(4.8') \qquad \frac{f-S_n(\Gamma+\hat{\varepsilon}_n)}{f-S_n(\Gamma)} \sim \frac{\Gamma(t-\varepsilon_{n+1}/\varepsilon_n) - \varepsilon_{n+1}}{1+\Gamma+t\Gamma}.$$

Let $z \in \tilde{D}_\alpha$ be kept fixed. Using (4.9) we also have that

$$(4.10) \qquad \frac{\mu_{n+2}}{\mu_{n+1}} = \frac{\varepsilon_{n+1}}{\varepsilon_n} \cdot \frac{1+\Gamma+\varepsilon_{n+2}+\Gamma\varepsilon_{n+2}/\varepsilon_{n+1}}{1+\Gamma+\varepsilon_{n+1}+\Gamma\varepsilon_{n+1}/\varepsilon_n}.$$

That is, $\{\Gamma\varepsilon_{n+1}/\varepsilon_n\}$ is the values of the tails of the convergent, limit periodic continued fraction $K(c_n/d_n)$ given by

$$(4.11) \qquad c_n = \Gamma(1+\Gamma+\varepsilon_n)\mu_{n+1}/\mu_n, \quad d_n = 1+\Gamma+\varepsilon_{n+1}-\Gamma\mu_{n+1}/\mu_n.$$

(See Section 2A.) This means that $\Gamma\varepsilon_{n+1}/\varepsilon_n \to \Gamma t$ (which we already know), and we can use Lemma 4.1 to estimate the speed. We get

$$(4.12) \qquad \left|\Gamma\frac{\varepsilon_{n+1}}{\varepsilon_n} - \Gamma t\right| < Q_n' \sim \frac{p_n+|\Gamma t|q_n}{|1+\Gamma|-|\Gamma t|}$$

where $p_n = \max_{m\geq n} |c_m - \Gamma(1+\Gamma)t| = \max_{m\geq n} |\alpha z(t-\mu_{m+1}/\mu_m) - \Gamma\varepsilon_m\mu_{m+1}/\mu_m| \sim$
$\max_{m\geq n} |\alpha z(t-\mu_{m+1}/\mu_m) - \Gamma\varepsilon_m t| \leq |\alpha z| \max_{m\geq n} |t-\mu_{m+1}/\mu_m| + |\Gamma t| \max_{m\geq n} |\varepsilon_m|$,
and $q_n = \max_{m\geq n} |d_m - (1+\Gamma-\Gamma t)| = \max_{m\geq n} |\varepsilon_{m+1} + \Gamma(t-\mu_{m+1}/\mu_m)| \lesssim \max_{m\geq n}|\varepsilon_{m+1}|$
$+ |\Gamma| \max_{m\geq n} |\bar{t}-\mu_{m+1}/\mu_m|$. From the proof of Theorem 3.1 follows moreover that $\max_{m\geq n} |\varepsilon_m| \leq R_n \sim \max_{m\geq n} |\mu_m z|/(|1+\Gamma|-|\Gamma|)$. This proves (3.5).

<u>PROOF of Theorem 3.3.</u> Let

$$\hat{\alpha}_n^* = (\Gamma^* + \hat{\varepsilon}_{n-1}^*)(1+\Gamma^*+\hat{\varepsilon}_n^*).$$

Then

$$|\alpha_n z - \hat{\alpha}_n^*(z)| = \left|\mu_n z - \frac{\Gamma^*\mu_{n+1}z + (1+\Gamma^*)\mu_n z + \mu_n\mu_{n+1}z^2/(1+\Gamma^*+t\Gamma^*)}{1+\Gamma^*+t\Gamma^*}\right|$$

$$\leq |\mu_n z|\left(\frac{|\Gamma^*||t-\mu_{n+1}/\mu_n|}{|1+\Gamma^*+t\Gamma^*|} + \frac{|\mu_{n+1}z|}{|1+\Gamma^*+t\Gamma^*|^2}\right)$$

$$\leq P|t'|^n(Qs'^{-n} + R|t'|^n) \leq P_1 r'^{-n},$$

where P, Q, R, P_1: $D^*_{\alpha,r}$ are properly chosen continuous functions and $t < |t'| < 1$, $s < s' < 1$. By use of [2, Theorem 4.1] follows then that Theorem 3.3 holds with r replaced by r'' such that $r' < r'' < 1$. Since t' and s' can be chosen arbitrarily close to t and s respectively, this proves the result.

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