CONSTRUCTIVE THEORY OF FUNCTIONS'87
SOFIA, 1988

GENERALIZED FRACTIONAL INTEGRAL AND FRACTIONAL DERIVATIVE REPRESENTATIONS OF HYPERGEOMETRIC FUNCTIONS ${}_pF_q$ FOR p=q OR p=q+1 *)

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1. Introduction. In [1] a generalized fractional calculus is proposed with some applications to different topics in analysis. Most of the known operators of generalized integration and differentiation are quite special cases of the generalized operators of integro-differentiation of fractional multiorder introduced there. In a series of papers we have discussed their different applications: to Bessel-type operators of Dimovski [2], [3] (see [4], [5]), to hyper-Bessel ordinary differential equations (see e.g. [5]), to special functions (see [6], [7], [8]), to generalized operators of integration and differentiation of Gelfond-Leontiev, to dual integral equations with Meijer's G-function as kernel, to Borel-Džrbasjan and Obrechkoff integral transforms.

Here special attention is devoted to the generalized hypergeometric functions ([9₁], p. 187):

$$(1)\quad {}_pF_q(a_1,\dots,a_p;b_1,\dots,b_q;z)=\sum_{k=0}^{\infty}\frac{(a_1)_k\dots(a_p)_k}{(b_1)_k\dots(b_q)_k}\frac{z^k}{k!}\ ,$$

where

$$(a)_0=1,\ (a)_k=a(a+1)\dots(a+k-1)=\frac{\Gamma(a+k)}{\Gamma(a)}\quad\text{and}\quad \begin{cases} z\in\mathbb{C} & \text{if } p\le q,\\ |z|<1 & \text{if } p=q+1\end{cases}\ .$$

*) Partially supported by the Council for Science in Bulgaria, grant no. 5/1987.

The following integral representation of Bessel functions $J_\nu(x)$ (Poisson's integral, [9_2], p. 14):

$$(2)\quad J_\nu(x)=\frac{2}{\sqrt{\pi}}\frac{(\frac{x}{2})^\nu}{\Gamma(\nu+\frac{1}{2})}\int_0^1 (1-t^2)^{\nu-\frac{1}{2}}\cos xt\ dt\ ,\ \nu>-\frac{1}{2}\ ,$$

and the differential representation of the spherical Bessel functions ([9_2], p. 73):

$$(3)\quad J_{-n-\frac{1}{2}}(x)=\frac{(2x)^{n+\frac{1}{2}}}{\sqrt{\pi}}\frac{d}{(dx^2)^n}\left\{\frac{\cos x}{x}\right\}\ ,\ n=0,1,2,\dots$$

are well known. Generalizations of them concerning ${}_pF_q$-functions in the case $p<q$ were proposed in [6], [7], [8]. As interesting particular cases of these results analogues of (2),(3) for the so-called hyper-Bessel functions of Delerue (1953): $J^{(m)}_{\nu_1,\dots,\nu_m}(x)$ were obtained. Now we propose representations analogous to (2),(3) for ${}_pF_q$-functions in the other two cases: p=q and p=q+1. For the sake of simplicity we deal only with a real variable x: $0\le x<\infty$, but these representations can be easily extended to the complex plane $|z|<\infty$ (for p=q), or to the unit disc $|z|<1$ (for p=q+1).

2. Generalized operators of fractional integration and differentiation. The kernel-function of our operators of fractional integration is a specific case of Meijer's G-function (see [9_1], p. 207):

$$G^{m,n}_{p,q}\left[\sigma\middle|\begin{matrix}a_1,\dots,a_p\\ b_1,\dots,b_q\end{matrix}\right]=\frac{1}{2\pi i}\int_{\mathcal{L}}\frac{\prod_{k=1}^{m}\Gamma(b_k-s)\prod_{j=1}^{n}\Gamma(1-a_j+s)}{\prod_{k=m+1}^{q}\Gamma(1-b_k+s)\prod_{j=n+1}^{p}\Gamma(a_j-s)}\sigma^s ds\ .$$

Definition. Let $m\geqslant 1$ be an integer, $\beta>0$, the "weights" $\gamma_1,\dots,\gamma_m$ be real numbers and $\delta_1\geqslant 0,\dots,\delta_m\geqslant 0$ be the components of the "multiorder of integration" $(\delta_1,\dots,\delta_m)$. Define the basic integral operators for functions of the space $(k\geqslant 0)$

$$C^{(k)}_\alpha=\left\{f(x)=x^p\tilde{f}(x);\ p>\alpha\ ,\ \tilde{f}\in C^{(k)}[0,\infty)\right\}\ ,\alpha\geqslant\max_k\left[-\beta(\gamma_k+1)\right]$$

by means of the following representation:

$$(4)\quad R_{\beta,m}^{(\gamma_k),(\delta_k)} f(x)=\int_0^1 G_{m,m}^{m,0}\left[\sigma \,\middle|\, \begin{matrix}(\gamma_k+\delta_k)\\(\gamma_k)\end{matrix}\right] f(x\sigma^{\frac{1}{\beta}})\, d\sigma \quad .$$

Then every operator of the form

$$(5)\quad R\, f(x) = x^{\beta\delta_0}\, R_{\beta,m}^{(\gamma_k),(\delta_k)} f(x)\ ,\quad \delta_0 \geqslant 0\ ,$$

is said to be a <u>generalized (m-dimensional) operator of integration of fractional multiorder ($\delta_1,\ldots,\delta_m$) of Riemann-Liouville (R.-L.)</u> type, or briefly: a <u>generalized fractional integral.</u>

When <u>m=1</u> the operator (4) coincides with the well-known <u>Erdélyi-Kober operator</u> of fractional integration:

$$(6)\quad R_{\beta,1}^{\gamma,\delta} f(x) = I_{\beta}^{\gamma,\delta} f(x) = \int_0^1 \frac{\sigma^{\gamma}(1-\sigma)^{\delta-1}}{\Gamma(\delta)}\, f(x\sigma^{\frac{1}{\beta}})\, d\sigma\ ,\quad \delta>0.$$

Special cases of (5) are: the <u>classical R.-L. fractional integral</u> :

$$(7)\quad R^{\delta} f(x) = x^{\delta}\int_0^1 \frac{(1-\sigma)^{\delta-1}}{\Gamma(\delta)}\, f(x\sigma)\, d\sigma \quad = x^{\delta}\, R_{1,1}^{0,\delta} f(x),$$

the operators of Hardy-Littlewood, Džrbasjan-Gelfond-Leontiev, etc. For <u>m=2</u> the operators (4) are the so-called <u>"hypergeometric fractional integrals"</u> (see papers of E.R.Love, Saxena and Kalla, M. Saigo):

$$R_{\beta,2}^{(\gamma_1,\gamma_2),(\delta_1,\delta_2)} f(x)=\int_0^1 \frac{\sigma^{\gamma_2}(1-\sigma)^{\delta_1+\delta_2-1}}{\Gamma(\delta_1+\delta_2)}\ {}_2F_1(\ldots;\ldots,1-\sigma) f(x\sigma^{\frac{1}{\beta}})\, d\sigma.$$

In the case <u>$m>2$</u> the operators (5) have not been investigated systematically except for the <u>Bessel-type operators of Dimovski</u> (see [2], [3], [4], [5]) and the corresponding <u>generalized Poisson-Sonine transmutations</u>.

In [1], [10], [11] some basic analytical properties of (5), a series of manipulating rules and inversion formulas are proposed with proofs and more details and examples. The following proposition is fundamental for many of the applications of the operators (5).

<u>Theorem</u>. ([1], [10], [11]) The operators (4) are <u>m-dimensional compositions</u> of commuting Erdélyi-Kober operators $I_{\beta}^{\gamma_k,\delta_k}$, $k=1,\ldots,m$, namely:

(8) $R_{\beta,m}^{(\gamma_k),(\delta_k)} f(x) = \left(\prod_{k=1}^{m} I_{\beta}^{\gamma_k,\delta_k}\right) f(x)$

$$= \int_0^1 \cdots \int_0^1 \left(\prod_{k=1}^{m} \frac{(1-\sigma_k)^{\delta_k-1}}{\Gamma(\delta_k)} \sigma_k^{\gamma_k}\right) f\left(x(\sigma_1 \dots \sigma_m)^{\frac{1}{\beta}}\right) d\sigma_1 \dots d\sigma_m.$$

The following two important properties of (4) (see [10], [11]):

(9) $R_{\beta,m}^{(\gamma_1,\dots,\gamma_m),(0,\dots,0)} = I$ - the identity operator,

(10) $R_{\beta,m}^{(\gamma_k+\delta_k),(\sigma_k)} R_{\beta,m}^{(\gamma_k),(\delta_k)} = R_{\beta,m}^{(\gamma_k),(\sigma_k+\delta_k)}$ (law of indeces)

are the analogues of the well-known ones of R.-L. fractional integral (7). They yield the following "formal" inversion formula:

$$\left\{R_{\beta,m}^{(\gamma_k),(\delta_k)}\right\}^{-1} f(x) = R_{\beta,m}^{(\gamma_k+\delta_k),(-\delta_k)} f(x).$$

To make this inversion formula a "correct" one, it is necessary to give a meaning to the symbols $R_{\beta,m}^{(\gamma_k'),(\delta_k')}$ for nonpositive multiorder ($\delta_1' \le 0,\dots,\delta_m' \le 0$). This leads to the following new

Definition*. Denote by

$$\eta_k = \begin{cases} [\delta_k]+1 & \text{for noninteger } \delta_k, \\ \delta_k & \text{for integer } \delta_k, \end{cases} \quad k=1,\dots,m.$$

The integro-differential operator of the form

(11) $D f(x) = D_{\beta,m}^{(\gamma_k),(\delta_k)} x^{-\beta\delta_0} f(x) = R_{\beta,m}^{(\gamma_k+\delta_k),(-\delta_k)} x^{-\beta\delta_0} f(x)$

$$= \left\{\left[\prod_{k=1}^{m} \prod_{j=1}^{\eta_k} \left(\frac{1}{\beta} x \frac{d}{dx} + \gamma_k + j\right)\right] R_{\beta,m}^{(\gamma_k+\delta_k),(\eta_k-\delta_k)}\right\} x^{-\beta\delta_0} f(x),$$

defined for functions $f \in C_\alpha^{(\eta_1+\dots+\eta_m)}$, $\alpha \geq \max_k\left[-\beta(\gamma_k+1)\right]$, is said to be a <u>generalized m-dimensional operator of fractional differentiation</u>, or: <u>generalized fractional derivative</u> of order $(\delta_1,\dots,\delta_m)$.

Further we shall need the following simple properties of the generalized fractional integrals and derivatives (see [10], [11]):

(12) $R_{\beta,m}^{(\gamma_k),(\delta_k)} x^{\beta\lambda} f(x) = x^{\beta\lambda} R_{\beta,m}^{(\gamma_k+\lambda),(\delta_k)} f(x),$

$$(12^*)\quad D^{(\gamma_k),(\delta_k)}_{\beta,m}\, x^{\beta\lambda} f(x) = x^{\beta\lambda} D^{(\gamma_k+\lambda),(\delta_k)}_{\beta,m} f(x)\ ,$$

$$(13)\quad R^{(\gamma'_k),(\delta'_k)}_{\beta,m_1} R^{(\gamma''_k),(\delta''_k)}_{\beta,m_2} f(x) = R^{((\gamma'_k),(\gamma''_k)),((\delta'_k),(\delta''_k))}_{\beta,m_1+m_2} f(x),$$

$$(13^*)\quad D^{(\gamma'_k),(\delta'_k)}_{\beta,m_1} D^{(\gamma''_k),(\delta''_k)}_{\beta,m_2} f(x) = D^{((\gamma'_k),(\gamma''_k)),((\delta'_k),(\delta''_k))}_{\beta,m_1+m_2} f(x).$$

3. Poisson-type integral representations of ${}_pF_q$-functions in the cases $p=q$ and $p=q+1$. It is known that the ${}_{p+1}F_{q+1}$-function can be represented as suitably chosen R.-L. fractional integral of a ${}_pF_q$-function, viz.

Lemma 1. ([12], p. 200) If $b_{q+1} > a_{p+1} > 0$ and $p \le q+1$, then

$$(14)\quad {}_{p+1}F_{q+1}\,(a_1,\dots,a_p,a_{p+1};b_1,\dots,b_q,b_{q+1};\ a_0x)$$
$$= x^{1-b_{q+1}} \frac{\Gamma(b_{q+1})}{\Gamma(a_{p+1})}\ R^{b_{q+1}-a_{p+1}}\left\{x^{a_{p+1}-1}\,{}_pF_q(a_1,\dots,a_p;b_1,\dots,b_q;a_0x)\right\}$$

(if $p=q+1$, then (14) is true for $|a_0x|<1$).

For $p=q=k$ this result can be rewritten in the form

$$(14')\quad {}_{k+1}F_{k+1}\,(a_1,\dots,a_{k+1};b_1,\dots,b_{k+1};\ a_0x)$$
$$= \frac{\Gamma(b_{k+1})}{\Gamma(a_{k+1})}\ R^{a_{k+1}-1,\,b_{k+1}-a_{k+1}}_{1,1}\left\{{}_kF_k(a_1,\dots,a_k;b_1,\dots,b_k;\ a_0x)\right\}\ .$$

Combining this representation with

Lemma 2. ([12], p. 187) If $b_1 > a_1 > 0$, then

$$(15)\quad {}_1F_1\,(a_1;b_1;\ a_0x) = \frac{\Gamma(b_1)}{\Gamma(a_1)}\, x^{1-b_1}\ R^{b_1-a_1}\, x^{a_1-1}\left\{e^{a_0x}\right\}$$
$$= \frac{\Gamma(b_1)}{\Gamma(a_1)}\ R^{a_1-1,\,b_1-a_1}_{1,1}\left\{e^{a_0x}\right\}\ ,$$

we are able to prove the following

Theorem 1. Let $p=q$ and $b_k > a_k > 0$, $k=1,\dots,p$. Then the "confluent" hypergeometric function ${}_pF_p\,(a_0x)$ can be represented as a generalized fractional integral of the function $\left\{x^{a_1-1}e^{a_0x}\right\}$:

$$(16)\quad {}_pF_p\,(a_1,\dots,a_p;b_1,\dots,b_p;a_0x) = \Gamma'.\ x^{1-a_1}\ R^{(\gamma_k),(\delta_k)}_{1,p}\left\{x^{a_1-1}e^{a_0x}\right\}$$

$$= \Gamma'' \int_0^1 G^{p,0}_{p,p}\left[\sigma \,\middle|\, \begin{matrix}(b_k)\\(a_k)\end{matrix}\right] \sigma^{-1} e^{a_0 x\sigma} d\sigma ,$$

where $\Gamma'' = \prod_{j=1}^{p} \frac{\Gamma(b_j)}{\Gamma(a_j)}$ and the parameters of the generalized fractional integral are: $\gamma_k = a_k - a_1$, $\delta_k = b_k - a_k$, $k=1,\dots,p$.

Proof. Applying (14') (p-1) times we obtain:

$${}_pF_p(a_0x) = \frac{\Gamma(b_p)}{\Gamma(a_p)} R^{a_p-1,b_p-a_p}_{1,1}\left\{ {}_{p-1}F_{p-1}(a_0x)\right\}$$

$$= \frac{\Gamma(b_p)\,\Gamma(b_{p-1})}{\Gamma(a_p)\,\Gamma(a_{p-1})} R^{a_p-1,b_p-a_p}_{1,1} R^{a_{p-1}-1,b_{p-1}-a_{p-1}}_{1,1}\left\{ {}_{p-2}F_{p-2}(a_0x)\right\}$$

$$= \dots = \prod_{j=2}^{p}\left[\frac{\Gamma(b_j)}{\Gamma(a_j)} R^{a_j-1,b_j-a_j}_{1,1}\right]\left\{ {}_1F_1(a_0x)\right\} .$$

According to Lemma 2 this expression is equal to

$$\prod_{j=2}^{p}\left[\frac{\Gamma(b_j)}{\Gamma(a_j)} R^{a_j-1,b_j-a_j}_{1,1}\right]\frac{\Gamma(b_1)}{\Gamma(a_1)} R^{a_1-1,b_1-a_1}_{1,1}\left\{e^{a_0x}\right\} .$$

But due to (8) this composition of one-dimensional fractional integrals (E.-K. operators) is a p-dimensional fractional integral :

$${}_pF_p(a_0x) = \left[\prod_{j=1}^{p}\frac{\Gamma(b_j)}{\Gamma(a_j)}\right] R^{(a_k-1),(b_k-a_k)}_{1,p}\left\{e^{a_0x}\right\}$$

$$= \left[\prod_{j=1}^{p}\frac{\Gamma(b_j)}{\Gamma(a_j)}\right] x^{1-a_1} R^{(a_k-a_1),(b_k-a_k)}_{1,p}\left\{x^{a_1-1} e^{a_0x}\right\} ,$$

taking into account property (12). Thus the theorem is proved.

Remark 1. The ${}_pF_p$-function is symmetric with respect to the parameters $a_1,\dots,a_l,\dots,a_p$ and therefore a_1 can be replaced in formula (16) by anyone of a_l's, $l=1,\dots,p$.

Remark 2. For Laguerre polynomials $L_n^{(\alpha)}(x) = \binom{n+\alpha}{n}{}_1F_1(-n;\alpha+1;x)$, $n=0,1,2,\dots$ such an integral representation is not valid because $a_1=-n<0$ and the condition of the theorem is not satisfied.

In the case p=q+1 instead of Lemma 2 we shall use

Lemma 3. ([12], p.186). If $b_1>a_2>0$ and $|x|<1$, then

(17) $${}_2F_1(a_1,a_2;b_1;x) = \frac{\Gamma(b_1)}{\Gamma(a_2)}\, x^{1-b_1}\, R^{b_1-a_2}\left\{x^{a_2-1}(1-x)^{-a_1}\right\},$$

or in our notations:

(17') $${}_2F_1(a_1,a_2;b_1;x) = \frac{\Gamma(b_1)}{\Gamma(a_2)}\, R^{a_2-1,\,b_1-a_2}_{1,1,}\left\{(1-x)^{-a_1}\right\}$$

$$= \frac{\Gamma(b_1)}{\Gamma(a_2)}\, x^{1-a_2}\, R^{0,\,b_1-a_2}_{1,1}\left\{x^{a_2-1}(1-x)^{-a_1}\right\}.$$

Combining (17') with the corollary

$${}_{(k+1)+1}F_{k+1}(a_1,\dots,a_{k+1},a_{k+2};b_1,\dots,b_{k+1};x)$$

$$= \frac{\Gamma(b_{k+1})}{\Gamma(a_{k+2})}\, R^{a_{k+2}-1,\,b_{k+1}-a_{k+2}}_{1,1}\left\{{}_{k+1}F_k(x)\right\}$$ **of Lemma 1, we obtain**

<u>Theorem 2.</u> If <u>$p=q+1$</u>, $b_k > a_{k+1} > 0$, $k=1,\dots,q$ and $|x| < 1$, then the "Gauss-type" hypergeometric function ${}_{q+1}F_q(x)$ is a q-dimensional fractional integral of the function $\left\{x^{a_2-1}(1-x)^{-a_1}\right\}$:

(18) $${}_{q+1}F_q(a_1,\dots,a_q,a_{q+1};b_1,\dots,b_q;x) = \Gamma''.\, x^{1-a_2}\, R^{(a_{k+1}-a_2),(b_k-a_{k+1})}_{1,q}$$

$$\left\{x^{a_2-1}(1-x)^{-a_1}\right\} = \Gamma'' \int_0^1 G^{q,0}_{q,q}\left[\sigma \,\middle|\, \begin{matrix}(b_k)\\(a_{k+1})\end{matrix}\right] \sigma^{-1}(1-x\sigma)^{-a_1}\, d\sigma,$$

where $$\Gamma'' = \prod_{j=1}^{q}\left[\Gamma(b_j)/\Gamma(a_{j+1})\right].$$

<u>Corollary.</u> For <u>$q=1$</u> (18) reduces to the well -known <u>Euler formula</u> ([9₁], p.76) serving **as** analytical continuation of Gauss function:

$${}_2F_1(a_1,a_2;b_1;x) = \frac{\Gamma(b_1)}{\Gamma(a_2)\,\Gamma(b_1-a_2)} \int_0^1 \frac{(1-t)^{b_1-a_2-1}\, t^{a_2-1}}{(1-xt)^{a_1}}\, dt .$$

By the same reasons the general formula (18) can be used for analytical continuation of ${}_{q+1}F_q(x)$ from the unit disc to the larger domain $|\arg(1-x)| < \pi$. Another form of this result can be written using repeated integrals:

(18') $${}_{q+1}F_q(x) = \prod_{j=1}^{q}\left[\Gamma(b_j)/\Gamma(a_{j+1})\,\Gamma(b_j-a_{j+1})\right]$$

$$\times\int_0^1 \cdots \int_0^1 \prod_{k=1}^{q} \left[(1-t_k)^{b_k-a_{k+1}-1} \quad t_k^{a_{k+1}-1}\right] (1-xt_1\ldots t_q)^{-a_1} dt_1\ldots dt_q .$$

4. Generalized fractional derivative representations of ${}_pF_q$ -functions in the cases $p=q$ and $p=q+1$. For some other conditions imposed on the parameters these integral representations of ${}_pF_q$-functions can be replaced by integro-differential ones using the corresponding fractional derivatives (11) and their properties. A hint for this is given in the paper of Lavoie, Osler and Tremblay [13].

Lemma 1*. ([13] , p.261) If $a_{p+1}>0$, $a_{p+1}>b_{q+1}$, then

$$ {}_{p+1}F_{q+1}((a_i)_{i=1}^{p},a_{p+1};(b_j)_{j=1}^{q},b_{q+1};a_0x)$$
$$=\frac{\Gamma(b_{q+1})}{\Gamma(a_{p+1})}\left(x^{1-b_{q+1}} D^{a_{p+1}-b_{q+1}} x^{a_{p+1}-1}\right) {}_pF_q((a_i)_{i=1}^{p};(b_j)_{j=1}^{q};a_0x)$$

(for $p=q+1$ the additional condition $|a_0x|<1$ is required), or in our notations:

$$ {}_{p+1}F_{q+1}(a_0x) = \frac{\Gamma(b_{q+1})}{\Gamma(a_{p+1})} \quad D_{1,1}^{b_{q+1}-1,a_{p+1}-b_{q+1}} \left\{ {}_pF_q(a_0x)\right\} .$$

Using Lemma 1* and propositions analogous to Lemmas 2,3 (see [13]) as well as the properties (12*),(13*), we can prove in a similar way the following two theorems.

Theorem 3. If $p=q$ and $a_k>0$, $a_k>b_k$, $k=1,\ldots,p$, then the "confluent" hypergeometric function ${}_pF_p(x)$ is a generalized p-dimensional derivative of the exponential function taken with possible power weight:

$$(19)\quad {}_pF_p((a_k);(b_k);a_0x) = \left[\prod_{j=1}^{p} \frac{\Gamma(b_j)}{\Gamma(a_j)}\right] D_{1,p}^{(b_k-1),(a_k-b_k)} \left\{ e^{a_0x}\right\}$$
$$= \left[\prod_{j=1}^{p} \frac{\Gamma(b_j)}{\Gamma(a_j)}\right] x^{1-a_l} D_{1,p}^{(b_k-a_l),(a_k-b_k)} \left\{ x^{a_l-1} e^{a_0x}\right\}, l=1,\ldots,p$$

Theorem 4. If $p=q+1$ and $a_k>0$, $a_k>b_k$, $k=1,\ldots,q$, then the "Gauss-type" hypergeometric function ${}_{q+1}F_q(x)$ is a q-dimensional fractional derivative of the function $\left\{x^{a_l-1}(1-x)^{-a_{q+1}}\right\}$, $l=1,\dots,q$:

(20) $${}_{q+1}F_q(a_1,\dots,a_q,a_{q+1};b_1,\dots,b_q;x) = \left[\prod_{j=1}^{q}\frac{\Gamma(b_j)}{\Gamma(a_j)}\right] D_{1,q}^{(b_k-1),(a_k-b_k)}\left\{(1-x)^{-a_{q+1}}\right\} = \dots$$

Corollaries. When all the differences $\eta_k=a_k-b_k$, $k=1,\dots,q$ are integer numbers, then the operators of fractional differentiation in (19),(20) are purely differential operators of order $(\eta_1+\dots+\eta_q)$, which are polynomials of the Euler differential operator $\delta_x = x\frac{d}{dx}$. By analogy with the spherical Bessel and hyper-Bessel functions, representable by repeated differentiation of elementary (trigonometric) functions (see [8], formula (15)), the functions

$${}_pF_p((a_k)_1^p;(b_k)_1^p;x),\ {}_{q+1}F_q((a_k)_1^{q+1};(b_k)_1^q;x)$$ with $b_k=a_k-\eta_k$, $k=1,\dots,p,q$, η_k – integer numbers,

will be called "spherical" generalized hypergeometric functions.

The generalized hypergeometric functions ${}_pF_q$ are some of the most commonly used special functions. Therefore the integral and the integro-differential representations for them proposed here and in [7] , [8] lead to many interesting and useful particular cases.

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