EXTENDED SHAPE PRESERVING APPROXIMATIONS

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1. Introduction. We say that approximation process defined by $\{L_n\}_{n=1}^{+}$ where L_n: $f \to L_n f$, preserves the shape of generating function if both f and $L_n f \in M$, where M is a set of functions characterized by some global "shape" as, for example, positivity, monotonicity or convexity (may be of higher order).

Probobly best konown result is that Bernstein polynomials

$$B_n f = \sum_{i=1}^{n} f(\frac{i}{n}) \binom{n}{i} x^i (1-x)^{n-i}, \quad x \in [0,1], \quad n \in N,$$

preserve convexity of all orders [9]. There are many other positive linear operators which have same property. Preservation of shape is closely related with so called variation diminishing property introduced by Polya and Schoenberg [10 , 12] and this is a consequence of total positivity [6] . In [14] , Tzimbalario establishes all continuous linear operators which preserve convexity with respect to ECT-system $(u_0, u_1, \ldots, u_n)$ for $n \geq 1$. Among others the following theorem has been proved by authors of [7]:

Theorem 1. For every u-monotone on [0,1] function f, its Bernstein polynomial $B_n f$ is also u-monotone for all n.

This theorem generalizes convexity preserving property of B_n since the class of starshaped funcitons (which obtains for $u(t)=t$) contain the class of convex functions (if they satisfy $f(0)=0$).

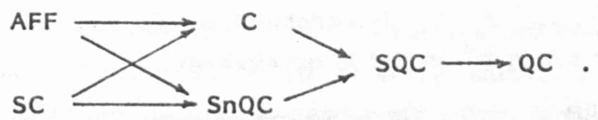
In this paper some further generalizations of convexity classes are discussed. Namely we shall deal with quasiconvexity, strictly quasiconvexity and strong quasiconvexity.

2. Variation diminshing operators and quasiconvexity. Let $f: I \to R$ $I \subset R$, $x, y \in I$, and $\mu = \theta x + (1-\theta) y$, $\theta \in (0,1)$. Then f belongs to

the class of quasiconvex (QC), strictly quasiconvex (SQC), strongly quasiconvex (SnQC), convex (C), strictly convex (SC) and affine (AFF) provided that for all $x, y \in I$

$$QC = \{ f : f(\mu) \leq \max(f(x), f(y)) \},$$
$$SQC = \{ f : f(\mu) < \max(f(x), f(y)),\ f(x) \neq f(y) \},$$
$$SnQC = \{ f : f(\mu) < \max(f(x), f(y)),\ x \neq y \},$$
$$C = \{ f : f(\mu) \leq \theta f(x) + (1-\theta) f(y) \},$$
$$SC = \{ f : f(\mu) < \theta f(x) + (1-\theta) f(y) \},$$
$$AFF = \{ f : f(\mu) = \theta f(x) + (1-\theta) f(y) \},$$

All these casses are important in nonlinear optimization. For more details see [1, 4] . In the sequel, we suppose f is bounded. In this case the classes SQC and SnQC are often call unimodal and strong unimodal class. The classes AFF, SC and C contain only continuous, functions, but SnQC, SQC and QC would not. So, if f is lower semicontinuos, we have following inclusions [11]



In the case when f is discontinuous, the class SQC is not included in QC as Karamardian's counterexample shows [4].

If is also known [1, 4, 11] that so called level set S(f,C) of f, defined as $S(f,C) = \{ (x,C) : x \in I,\ f(x) \leq C \}$, C is real constant, is convex for every C if and only if f is quasiconvex.

Further, let $v(\{a_i\}_{i=1}^{m})$ denotes the number of strict sign changes of the sequence $\{a_i\}_{i=1}^{m}$, i.e.the number of sign changes when zero terms of the sequence are neglect, and let $V(f) = \sup v(\{f(x_i)\}_{i=1}^{m})$ where supremum is taken over all sequences $\{x_i\}_{i=1}^{m}$ (from I) and all m. We say that approximation method, defined by the sequence $\{L_n\}_{n=1}^{+\infty}$ is variation diminshing if $V(L_n f) \leq V(f)$ for all n. Now, we shall prove

Theorem 2. Let $\{L_n\}_{n=1}^{+\infty}$ be a variation diminishing method so that $L_n f$ is continous and defined for every bounded function f. Then if $f \in QC$ then $L_n f \in QC$ or $-L_n f \in QC$ for every n.

Proof. Let $y_c(x)=C$, $x \in R$ and $F(x) = L_n f(x)$. Then, it is obvious $0 \leq V(f - y_c) \leq 2$ for every $f \in QC$ and every $C \in R$. By virtue of the variation diminishing property of L_n, we have $0 \leq V(F - y_c) \leq 2$ but, also $0 \leq V(-F - y_c) \leq 2$, as a consequence of $V(F) = V(-F)$. Let $C_j = \{C \in R : V(F - y_c) = j\}$ j = 0,1 and 2. Then by continuity of $L_n f$ we have $C_0 \cup C_1 \cup C_2 = R$, $C_i \cap C_j = \emptyset$, $i \neq j$, $i,j \in \{0, 1, 2\}$, Then, we have or $\inf C_1 = \sup C_2$ or $\inf C_2 = \sup C_1$. Suppose that we have $\inf C_1 = \sup C_2$. Then, following three cases can occur:

i) $C_1, C_2 = \emptyset$; This means that for all $C \in R$, $V(F - y_c) = 0$, i.e. $L_n f = F$ is a constant.

ii) $C_1 \neq \emptyset$, $C_2 = \emptyset$; In this case $V(F - y_c) = 1$, $C \in R$. For every C', $C'' \in R$, $C' \neq C''$ there exist unique x' and x'' ($x' \neq x''$) so that $F(x') = C'$ and $F(x'') = C''$. Then $(x' - x'')(F(x') - F(x''))$ have the constant sign, i.e. $F = L_n f$ is monotone, so $S(L_n f, C)$ is a half-line which is convex set in R^2 so $L_n f$ is quasiconvex.

iii) $C_2 \neq \emptyset$; Then, we can determine $p = \inf C_2$ and $q = \sup C_2$, and for every $C \in (p, q)$, $V(F - y_c) = 2$. This means that the graph of $F = L_n f$ crosses the line $y_c = C$ twice when $p < C < q$ in $x = x'$ and $x = x''$, so we have $F(x') = F(x'')$. The line segment which joins the points $(x', F(x'))$ and $(x'', F(x''))$ is the level set. To prove this assertion it is enough to prove that $F(x) \leq F(x')$ for all $x \in [x', x'']$. Suppose that it is not true, namely that there exists $t \in [x', x'']$ so that $F(t) > F(x')$. Then, the graph of F crosses the line $y = F(x')$ in two additional points in the neighbourhood of the point $(t, F(t))$, which contradicts to the fact that number of crosses on $[x', x'']$ is only 2. Thus, $S(f,C)$ is level set for all $C \in (p, Q)$ and it is convex. For $C < p$, $V(L_n f - y_c) = 0$, and for $C > q$ it is either 0 or 1, and only this last case is interesting for us. Namely, then we have the case ii).

In the case when $\inf C_2 = \sup C_1$, we can put $F = -L_n f$, and then $-L_n f \in QC$ is valid.

3. Some positive linear operators. Theorem 2 gives the positive answer to authors previous conjecture [8] that Bernstein polynomials preserve unimodality. This problem is connected with so called approximation extremum in nonlinear programming [1, 3]. Namely, the following theorem, more refined then Theorem 2 is valid for Bernstein polynomials:

Theorem 3. Let f be bounded on $[0, 1]$. Then, for $n \geq 2$ the following is valid:

1^o $f \in QC \Rightarrow B_n f \in SnQC$,

2^o $f \in SQC \Rightarrow B_n f \in SnQC$,

3^o $f \in C \Rightarrow B_n f \in SC$,

4^o $f \in AFF \Rightarrow B_n f \in AFF$.

Proof. Note that if f is bounded and quasiconvex it can have only discontinuities of the first kind, and then formula (2), p. 27 from [9] takes place. If f is convex then it must be continuous inside the definition interval.

The points 3^o and 4^o have been already known and we mention them here only for completness. The points 1^o or 2^o can be regarded as a consequence of Theorem 2. Namely, $B_n f \in QC$ for every mounded f from QC, and being quasiconvex $B_n f$ needs not havean inique minima in $[0, 1]$. But if we denote by $Z\{p_n(x)\}$ the number of nonnegative zeros of polynomial p_n, and apply Schoenberg's method [12], we have

$$\underset{(0,1)}{Z}\left\{ B_n{}'f(x) \right\} = \underset{(0,1)}{Z}\left\{ \frac{B_n{}'f(x)}{(1-x)^{n-1}} \right\} = \underset{(0,+\infty)}{Z}\left\{ n \sum_{i=0}^{n} \Delta f(\tfrac{i}{n}) \binom{n-1}{i} y^i \right\}$$

$\leq v\{ n\binom{n-1}{i} \Delta f(\frac{i}{n}) \}$ by Decartes' rule of signs, where we put $\Delta f(\frac{i}{n}) = f(\frac{i+1}{n}) - f(\frac{i}{n})$ and $y = x/(1-x)$. Furher, we have $v\{n\binom{n-1}{i} \Delta f(\frac{i}{n})\} = v(s_i)$, where $s_i = n \Delta f(\frac{i}{n})$ is the i-th slope of the polygonal line with vertices $(\frac{i}{n}, f(\frac{i}{n}))$, and for $f \in QC$ we have $v(s_i) = 0$ if f is monotone or $v(s_i) = 1$ if f changes monotonicity on $[0,1]$. Thus, ve have $v(s_i) \leq 1$, so the number of zeros of $B_n{}'f$ in (0,1) is exactely 1, i.e. $B_n f$ has unique minimum value in (0,1) and its graph contains no linear segments. This means tht $B_n f$ is strongly quasiconvex (strictly unimodal) for every quasiconvex f, and $n \geq 2$.

Consequence of Theorem 3. Consider the nonlinear optimization problem min f(x), $x \in [0, 1]$, where f(x) is bounded quasiconvex function. There are a number of numerical methods for solving this problem [1, 3, 4]. But we have, by Theorem 3 that the solution of above problem x^* can be approximated by $x^o = \min B_n f$, i.e. the nonlinear optimization problem can be reduced to solving of n-th order algebraic equation $B_n' f(x) = 0$.

The similar considerations are valid for Jakimovski-Levatan operators (see [14])

$$P_n f(x) = \frac{e^{-nx}}{g(1)} \sum_{i=0}^{+\infty} p_i(nx)\, f(\frac{i}{n}),\ x \in [0,+\infty),\ n \in N,$$

which generalized well known Szasz-Mirakyan operators. Of course, $P_n f$ is a polynomial, so instead $Z(P_n f)$ we can use only $V(P_n f)$.

4. Variation diminshing splines . Bernstein polynomial operators can be generalized in the sense of splines. Namely, the following operators has been introduced by Schoenberg[13]:

$$Sf(x) = \sum_{i=0}^{k-1} f(x_i)\, N_i(x) \quad , \quad f \in C\,[0,1] \ ,$$

where $N_i (i=0,\ldots, k-1)$ are normalized cubic B-splines defined on the mesh $t_o < t_1 < \ldots < t_{k+3}$ $(k > 3)$. The function Sf(x) is C^2 piecewise cubic curve which is called design curve and plays an important role in computer-aided design. The set of points $(x_i, f(x_i))$ then calls the control polygon. In [2] Barsky replaced B-splines with more general basic splines, so called β- splines, which makes the designed curve more flexible. It is known that B-splines are variation diminishing, i.e.

$$V\,(\sum_{i=0}^{k-1} a_i N_i) \leq v\,(\{a_i\}_{i=0}^{k-1})\ .$$

The same holds also for β - splines as Goodman showed in [5], even for generalized form of β- splines. Also in[5] we find that design curve is convex whenever control polygon is convex. By our Theorem 2 we have that design curve is quasiconvex when control polygon is quasiconvex, i.e. when f∈QC.

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