CONSTRUCTIVE THEORY OF FUNCTIONS'87

SOFIA, 1988

APPLICATIONS OF OPERATOR APPROXIMATION TO FOURIER ANALYSIS

M. A. Kon*, A. G. Ramm, and L. A. Raphael**

In this paper, we will consider the problem of when two weighted eigenfunction expansions approximate a function in the "same" way. We will investigate this from the standpoint of approximate operator methods in a function space setting. Versions of this problem have been examined in work studying the relationship between approximations by Fourier series and by Sturm-Liouville series. This began with the papers of Stone [St] on the relation of Fourier series to other one dimensional eigenfunction expansions. Benzinger extended this in work relating Riesz typical means of Stone-regular expansions to those of Fourier series.

It is known that some Sturm-Liouville expansions are not equiconvergent with (i.e., do not converge in the same way at a given point as) Fourier series. This changes, however, when Riesz typical means are used as an approximation method, wherein the two expansions do converge at the same points (i.e., they are equisummable). Questions of this sort have in fact been considered classically as well. The study of equiconvergence between Fourier series and series of eigenfunctions (such as Sturm-Liouville expansions) is contained in [Ha,Bi,Wa,Ta]. Equisummability in the same context has been studied in [St,Ti,LS,Be]. Recently Komornik [K] proved a general equiconvergence theorem for Schrödinger operators on one dimensional intervals. His result allows a complex potential, and is independent of boundary conditions.

We will apply general function space results to relate approximations using eigenfunctions of Schrödinger operators and of the Laplacian on general domains of $\mathbf{R}^n$ (on all of $\mathbf{R}^n$, Laplacian expansions reduce to the Fourier transform). In studying abstract summation, we require the Dunford analytic operator calculus. We note that analytic methods for multidimensional problems are important, in part, because ordinary convergence [Fe] and even Riesz convergence [KST] fail for ordinary multidimensional Fourier series in some L^p spaces. The former, in fact, fails in all L^p except L^2. We will give general theorems on equisummability for pairs of operators in Banach space, and illustrate them in the Schrödinger case.

We now illustrate the relationship of the present operator theoretic approach to ordinary

* Research partially supported by N.S.F. Grant DMS-8509458

** Research supported by Army Research Office grant number DAAG-84-G 000 4

approximation by eigenfunctions. Let G be a domain of $\mathbf{R}^n$. Let A be an operator on $L^p(G)$, with (possibly generalized) eigenfunctions $u(x,\lambda)$ and spectral values λ, with associated spectral expansion

$$f(x) \sim \int_{\sigma(A)} F_A(\lambda)u(x,\lambda)d\rho_A(\lambda). \tag{1}$$

Here $F_A(\lambda)$ denotes a generalized Fourier coefficient, and ρ is the spectral measure. Let $\mathbf{C}$ denote the complex numbers, and $\phi: \mathbf{C} \to \mathbf{C}$ be continuous, with $\phi(0) = 1$. Then in some topologies f is better approximated by the limit

$$f(x) = \lim_{\epsilon \to 0^+} \int_{\sigma(A)} F_A(\lambda)\phi(\epsilon\lambda)u(x,\lambda)d\rho_A(\lambda) \tag{2}$$

If B is an operator with eigenfunctions $v(x,\mu)$ and eigenvalues μ, our goal is to prove that if $f(x) \sim \int_{\sigma(B)} F_B(\mu)v(x,\mu)d\rho_B(\mu)$, then $\int_{\sigma(B)} F_B(\mu)\phi(\epsilon\mu)v(x,\mu)d\rho_B(\mu)$ converges to $f(x)$ as $\epsilon \to 0^+$, either at a given point or in L^p, if and only if (2) does. Suppose that $\phi(A)$ is defined for an operator A by an operator calculus. Then this problem can be reformulated as a question comparing approximations defined by analytic multipliers of operators, that is, when does convergence of $\phi(\epsilon A)f(x)$ to $f(x)$ as $\epsilon \to 0^+$ imply the same for $\phi(\epsilon B)f(x)$? We will study this first when $\phi(A)$ is the resolvent and later for other ϕ. Our abstract approach is useful because, in the general context of elliptic operators, it reduces the question of equisummability to easier and better-known estimates for elliptic operators.

I. Equisummation in Banach Space.

Here X and Y will be complex Banach spaces, with $X \cap Y$ dense in X. Let A and T be closed linear operators in X. Let the domain $\mathcal{D}(T) \supset \mathcal{D}(A)$, and $B = A + T$. We assume $\mathcal{D}(B) = \mathcal{D}(A)$. Let the spectrum $\sigma(A) \subset \mathbf{C}$, and the resolvent set be denoted by $\rho(A) = \mathbf{C}\backslash\sigma(A)$ with $\Sigma \subset (\rho(A) \cap \rho(B))$, $\Sigma \neq \phi$. The operators A and B are *resolvent equisummable* from X to Y (in Σ) if for $f \in X$

$$\|\zeta((\zeta I - A)^{-1}f - \zeta((\zeta I - B)^{-1}f\|_Y \to 0 \quad \text{as } |\zeta| \to \infty \text{ in } \Sigma. \tag{3}$$

We assume that fractional powers $(\zeta I - A)^\alpha$, $\alpha \in \mathbf{R}$ are defined for sufficiently large ζ in a sector of $\mathbf{C}$. This will be the case if the following maximum decrease condition $(\mathcal{A})$ holds:

$$\rho(A) \supset \Sigma_{\gamma,r} \equiv \{\zeta = \rho e^{i\theta}: \gamma < |\theta| \leq \pi,\ \rho > r\},$$

and $\|(\zeta I - A)^{-1}\| \leq C(1 + |\zeta|)^{-1}$ for $\zeta \in \Sigma_{\gamma,r}$. Let $\|\cdot\|_{X,Y}$ denote the operator norm from X to Y.

Let A, B be as above, with $T = B - A$. We will need:

Proposition 1: *Let $\Sigma_{\gamma,r} \subset \mathbf{C}$ and condition $(\mathcal{A})$ hold for A and B. If for some $0 < \alpha < 1$, $\|(\zeta I - A)^{-\frac{\alpha}{2}}T(\zeta I - A)^{-\frac{\alpha}{2}}\|_{X,X}$ is bounded for $|\zeta|$ sufficiently large, $\zeta \in \Sigma_{\gamma,r}$, then A and B are resolvent equisummable in $\Sigma_{\gamma,r}$.*

The proof of the proposition is based on estimates which rely on the identity

$$(A-\zeta I)^a = \frac{\sin \pi a}{\pi} \int_0^\infty \lambda^a (\lambda I + A - \zeta I)^{-1} d\lambda, \qquad (-1 < a < 0,\ \zeta \in \Sigma_{\gamma,r}), \tag{4}$$

which holds for $\zeta \in \Sigma_{\gamma,r}$ sufficiently large.

We also need

Proposition 2: *Let A, B, satisfy condition $(\mathcal{A})$, and T, $\Sigma_{\gamma,r}$ be as above, with $\rho(A) \cap \rho(B) \supset \Sigma_{\gamma,r}$. Assume that for some $a, c > 0$ and $\delta > 0$, with $c > 1-\delta$, we have*

$$\|TA^{-a}\|_{X,X} < \infty, \ \|(\zeta I - A)^{-1}\|_{X,Y} = O(|\zeta|^{-c})$$

and $\|A^a(\zeta I - A)^{-1}\|_{X,X} = O(|\zeta|^{-\delta})$ as $|\zeta| \to \infty$, $\zeta \in \Sigma_{\gamma,r}$. *Then*

$$\|\zeta(\zeta I - B)^{-1} - \zeta(\zeta I - A)^{-1}\|_{X,Y} = O\left(\zeta^{-\delta-c+1}\right) \qquad (|\zeta| \to \infty,\ \zeta \in \Sigma_{\gamma,r}). \tag{5}$$

Proof: We have

$$\zeta(\zeta I - B)^{-1} - \zeta(\zeta I - A)^{-1} = \zeta(\zeta I - B)^{-1} T (\zeta I - A)^{-1} \qquad (\zeta \in \Sigma_{\gamma,r}).$$

Thus

$$\begin{aligned} \|\zeta(\zeta I - B)^{-1} - \zeta(\zeta I - A)^{-1}\|_{X,Y} &\le \|\zeta(\zeta I - B)^{-1}\|_{X,Y} \|T(\zeta I - A)^{-1}\|_{X,X} \\ &\le \|\zeta(\zeta I - A)^{-1}\|_{X,Y} \|(I - T(\zeta I - A)^{-1})^{-1}\|_{X,X} \|T(\zeta I - A)^{-1}\|_{X,X}. \end{aligned} \tag{6}$$

Note that

$$\|T(\zeta I - A)^{-1}\|_{X,X} \le \|TA^{-a}\|_{X,X} \|A^a(\zeta I - A)^{-1}\|_{X,X} = O(|\zeta|^{-\delta}) \qquad (\zeta \in \Sigma_{\gamma,r}). \tag{7}$$

Equation (5) follows from (6) and (7).

Proposition 2 is important in applications to approximation by eigenfunctions of elliptic operators on manifolds; its hypotheses are often known or easily verified. The proposition deals directly with the use of the resolvent summation method to approximate functions, but can be extended to deal with summability for general analytic functions of A through the role of the resolvent in the Dunford operator calculus. Let X be a Banach space and $A: X \to X$ be linear with spectrum $\sigma(A) \subset \Omega$, with Ω a simply connected domain in $\mathbf{C}$. Let ϕ be an analytic function on Ω and $\Gamma \subset \Omega$ be a positively oriented contour in Ω enclosing $\sigma(A)$. We define

$$\phi(A) = \frac{1}{2\pi i} \int_\Gamma (\zeta I - A)^{-1} \phi(\zeta) d\zeta, \tag{8}$$

if the integral converges absolutely. We assume henceforth that $0 \in \Omega$, $\phi(0) = 1$, and that $\phi|_\Omega$ is bounded.

Let Y be a Banach space, with $X \cap Y$ dense in X, and let A, B be operators in X. We assume that the domain Ω contains 0 and the spectra $\sigma(A)$, $\sigma(B)$.

Definition: A and B are ϕ-equisummable from X to Y if $\|\phi(\epsilon B) - \phi(\epsilon A)\|_{X,Y} \to 0$ as $\epsilon \to 0^+$.

Proposition 3: *Under the above hypotheses and those of Proposition 2, assume that Ω contains a translate $S' = b + S$ $(b \in \mathbf{C})$ of a sector $S = \{\zeta : \theta_1 \leq \arg\zeta \leq \theta_2\} \subset \mathbf{C}$, with $0 \in S'$, $\sigma(A) \subset S'$ and $\partial S' \subset \Sigma_{\gamma,r}$. Then A and B are ϕ-equisummable from X to Y in $\Sigma = \sim \Omega$.*

Proof: The proof follows by the insertion of the bounds in Proposition 2 into the formula given by (8) for the difference $\phi(\epsilon A) - \phi(\epsilon B)$, and taking the limit $\epsilon \to 0$.

II. An Application to Schrödinger Theory

Let $G \subset \mathbf{R}^n$ be a (possibly unbounded) domain with C^∞ boundary ω. We assume for regularity that ω has bounded curvature, and uniformly bounded $(n-1)$-Lebesgue measure in any unit ball $B \subset \mathbf{R}^n$. We define the Sobolev space $H^s(\mathbf{R}^n) = \{f : (1+|\xi|)^s \hat{f}(\xi) \in L^2(\mathbf{R}^n)\}$, where $\hat{}$ denotes Fourier transform. Let $H^s(G)$ consist of those $u \in L^2(G)$ which have extensions $\tilde{u} \in H^s(\mathbf{R}^n)$, $\tilde{u}|_G = u$. If $u \in H^s(G)$, define $\|u\|_{H^s(G)} = \|u\|_s = \min\{\|\tilde{u}\|_{H^s(\mathbf{R}^n)} : \tilde{u}|_G = u\}$.

Let $A = -\Delta$ and $B = A + T$ be a Schrödinger operator

$$B = -\Delta + \vec{b}(x) \cdot \vec{\nabla} + b_0(x)$$

with a vector potential term, $\vec{b}(x) = (b_1(x), \ldots b_n(x)) \in L^{n+\epsilon} + L^\infty$ and potential $b_0 \in L^{\frac{n}{2}+\epsilon} + L^\infty$ for some $\epsilon > 0$. By $L^r + L^\infty$ we mean $\{f = f_1 + f_2 : f_1 \in L^r, \ f_2 \in L^\infty\}$. More precisely, let $b_i(x) \in L^{r_i}(G) + L^\infty(G)$, with $r_i > n$ $(i \geq 1)$ and $r_0 > \frac{n}{2}$. Define

$$d \equiv \max\left(\sup_{i\geq 1} \frac{n}{r_i} + 1, \ \frac{n}{r_0}\right); \tag{9}$$

note $d < 2$. This is essentially a measure of the singularity of the coefficients of B. The domain $\mathcal{D}(A)$ is defined by closing the action of A on functions $u \in H^2(G)$ which satisfy the homogeneous boundary condition

$$u(x)\cos\theta + \partial_n u(x)\sin\theta = 0, \qquad \left(x \in \omega, \ 0 \leq \theta \leq \frac{\pi}{2}\right). \tag{10}$$

Then we have:

Theorem 1: *If $B = -\Delta + \vec{b}(x) \cdot \vec{\nabla} + b_0(x)$ is the Schrödinger operator above, with boundary condition (10), then B and $A = -\Delta$ are resolvent equisummable from L^p to L^q in a region of the* *form* $\Sigma_{\gamma,r}$, *for any* $\gamma > 0$, *and* $r = r(\gamma)$, *if* $\alpha = \frac{n}{2}\left(\frac{1}{p} - \frac{1}{q}\right) + \frac{d}{2} - 1 < 0$, *d as in (9). In this case, the rate of convergence is given by*

$$\|\zeta(\zeta I - A)^{-1} - \zeta(\zeta I - B)^{-1}\|_{p,q} = O(|\zeta|^{\alpha}). \tag{11}$$

We remark that setting $q = \infty$ gives conditions here for pointwise equisummability. Since the eigenfunction expansion with respect to the Laplacian in $\mathbf{R}^n$ is essentially the Fourier transform, Theorem 1 relates approximations by Fourier integrals to approximations by Schrödinger eigenfunctions. The same can be done for ϕ-summability using Theorem 2.

Proof of Theorem 1: The proof follows through estimates on the resolvent kernel of the Laplacian in [Ko], and with the help of (4), resulting estimates on fractional powers of the resolvent. Finally, in Proposition 2 we let $A = I - \Delta$ and $B = -\Delta + \vec{b}(x) \cdot \vec{\nabla} + b_0$, and use estimates on (L^p, L^q) norms of the operators $T(I-\Delta)^{-a}$, $(I-\Delta)^{a}(\zeta I - I + \Delta)^{-1}$, and $((\zeta I - I + \Delta)^{-1}$ (along with z-derivatives of the latter, to obtain the hypotheses of the proposition.

We remark that the estimates used above are known (in various forms) for elliptic operators on other spaces. In addition, the condition on α in Theorem 1 is sharp.

Applying Proposition 3 to Theorem 1, we have

Theorem 2: *If B is as in Theorem 1 and* $\phi(\zeta)$ *is bounded and analytic in a domain* $-a + S = \{-a + \zeta\colon\ \zeta \in S\}$, *where S is a sector containing* $\mathbf{R}^+$, $a > 0$, *and* $\phi(0) = 1$, *then B and* $-\Delta$ *are* ϕ*-equisummable from* L^p *to* L^q *in* $\Sigma = \{\zeta :\ \arg\zeta \in (-\theta, \theta)\}$ *for all* $0 < \theta < \pi$, *if*

$$\frac{1}{p} - \frac{1}{q} < \frac{2-d}{n}, \tag{12}$$

where d is given by (9).

The semigroup $\phi(B) = e^{-Bt}$ is perhaps the most important function to which the above applies; in this context Theorem 2 is a statement that the solution of the perturbed heat equation $\frac{\partial}{\partial t}u(x,t) = -Bu(x,t)$ where $B = -\Delta + q(x)$ converges to its L^p initial values at exactly the same points as that of the unperturbed equation; in particular, convergence as $t \to 0$ is almost everywhere.

III. Operators with Growing Coefficients

We finish by giving an application of the present approach to approximation by eigenfunctions of operators with coefficients growing at ∞.

Theorem 3: *Let* $q(x)$ *be a real-valued function on* $\mathbf{R}^n$ *which is bounded from below, with* $q(x) \to \infty$ *as* $|x| \to \infty$, *and* $|r(x)| \le C|q(x)|^a$, $a < 1$ *for x sufficiently large, where* $r \in L^{\frac{n}{2}+\epsilon}_{loc}$ *is complex valued and* $\epsilon > 0$. *Then the operators* $-\Delta + q$ *and* $-\Delta + q + r$ *are resolvent equisummable in* L^2 *on* $\Sigma_{\gamma,r} \equiv \mathbf{R}^- \subset \mathbf{C}$, where $\mathbf{R}^-$ denotes the negative reals.

Proof: Without loss, we may assume for the proof that $q > 0$. The proof relies on a series of technical lemmas, aimed first at showing that the resolvent kernel of the full operator is positive, and then to bound the L^2- operator norm of HrH, where $H = (\lambda - \Delta + q))^{-\frac{\alpha}{2}}, \quad 0 < \alpha < 1$. Finally, Proposition 1 is applied to this operator product to complete the proof.

References

[Be] Benzinger, H., Green's function for ordinary differential operators, J. Differential Equations 7 (1970), 478-496.

[Bi] Birkhoff, G.D., Boundary value and expansion problems of ordinary linear differential equations, Transactions A.M.S. 9 (1908), 373-395.

[Fe] Fefferman, C., The new multiplier theorem for the ball, Ann. Math. 94 (1971), 330-336.

[GK] Gurarie, D. and Kon M., Radial bounds for perturbations of elliptic operators. J. Functional Analysis 56 (1984), 99-123.

[Ha] Haar, A., Zur theorie der orthogonalen Funktionensysteme, Math. Annalen 69 (1910), 331-371.

[KST] Kenig C.E., R.J. Stanton, and P.A. Tomas, Divergence of eigenfunction expansions, J. Functional Analysis 46 (1982), 28-44.

[K] Komornik, V., An equiconvergence theorem for the Schrödinger operator, Acta Math. Hungar. 44 (1984), 101-114.

[Ko] Kon, M., Regularity properties of Schrödinger operators on a domain of $\mathbf{R}^n$, Differential Equations, I.W. Knowles and R.T. Lewis, eds., 359-366, Elsevier Science Publishers, 1984.

[LS] Levitan, B., and Sargsjan, I., Introduction to the Spectral Theory; Self-Adjoint Differential Operators, A.M.S. Translations, Providence, 1975.

[Ra] Raphael, L.A., Equisummability of eigenfunction expansions under analytic multipliers, J. Math. Analysis and Applications 115 (1986), 93-104.

[St] Stone, M.H., A comparison of the series of Fourier and Birkhoff, Transactions A.M.S. 28 (1926), 695-761.

[Ta] Tamarkin, J.D., Sur quelques points de la theorie des equations differentielles lineaires ordinaires et sur la generalisation de la serie de Fourier, Rendiconti del Circolo Mathematico di Palermo 34 (1912), 345-395.

[Ti] Titchmarsh, E.C., Eigenfunction Expansions, parts I and II, Oxford, 1962.

[Wa] Walsh, J.L., On the convergence of the Sturm-Liouville series, Ann. Math. 24 (1922-23), 109-120.

Columbia University, New York, NY and Boston University, Boston, MA, U.S.A.

Kansas State University, Manhattan, KS, U.S.A.

Howard University and the National Science Foundation, Washington, D.C., U.S.A.