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ON THE DROP PROPERTY OF CONVEX SETS IN BANACH SPACES

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1. Introduction. Inspired by the articles of Daneš [1] and J.-P. Penot [6], S. Rolewicz [7] introduced the notion of drop property of a norm (more precisely, of the unit ball) in a Banach space. He proved that if a norm is uniformly rotund then it has the drop property and the drop property of a norm implies reflexivity of the space. Later it was proved in [4] that if a norm is 2-rotund then it has the drop property and if a norm is rotund and has the drop property then it is weakly 2-rotund. A norm of a Banach space X is said to have the property (H) if the conditions $\{x_n\}\subset X$, $\{x_n\}$ converges weakly to an element $x\in X$ and $\|x_n\|\to\|x\|$ imply that $x_n\to x$ in the norm topology. Recently, V. Montesinos [5] proved that up to equivalent norm the drop property of the unit ball gives a characterization of reflexivity. More precisely, he showed that a norm of a Banach space X has the drop property iff it has the property (H) and X is reflexive (i.e. iff X is a space of Efimov - Stechkin).

In the present paper instead of the unit ball we consider the general case of a convex, closed and bounded set. Let $(X,\|\cdot\|)$ be a Banach space and C be a convex, closed and bounded subset of X. By the drop $D(x,C)$ determined by a point $x\in X$, $x\notin C$ we shall mean the convex hull of the set $\{x\}\cup C$. C is said to have the drop property if for every closed set A, disjoint with C, there exists an element $a\in A$ such that $D(a,C)\cap A=\{a\}$. It is easy to show that a similar result to that of Montesinos holds for sets with non-empty interior but it could not be extended in the general case. We prove that a covex, closed and bounded set in a non-reflexive Banach space has the drop property iff it is compact. For reflexive Banach spaces we give a characterization of the drop property for the case of symmetric sets - such a set has the drop property iff it is either compact or has non-empty interior and every support point is a point of continuity.

2. Results. We can prove as in [7] the following statement.

Proposition 2.1. Let C be a convex, closed and bounded subset of a Banach space X. Let there exist an $f \in X^*$, $f \neq 0$ such that

$$\inf \{ \alpha(S_\varepsilon) : \varepsilon > 0 \} > 0 ,$$

where $S_\varepsilon = \{ x \in C : f(x) \geqq M - \varepsilon \}$ and $M = \sup \{ f(x) : x \in C \}$. Then the set C does not have the drop property.

The Kuratowski´s index of non-compactness of a set A, $\alpha(A)$, is the infimum of all positive numbers r such that A can be covered by a finite number of sets of diameter less than r.

Utilizing the above Proposition we have

Proposition 2.2. Let X be a Banach space and C be a convex, closed and bounded subset of X with the drop property. Then every support point x of C (i.e. for which there is an $f \in X^*$, $f \neq 0$, sothat $f(x) = \sup \{ f(y) : y \in C \}$) is a point of continuity (i.e. $y_n \to x$ in the norm topology whenever $\{y_n\} \subset C$, $\{y_n\}$ tends weakly to x).

By Proposition 2.1 and a theorem of James we obtain

Proposition 2.3. Let X be a Banach space and C be a convex, closed and bounded subset of X with the drop property. Then C is weakly compact.

Conversely, the weak compactness and the continuity property of the support points of C imply that for every functional $f \in X^*$, $f \neq 0$,

$$\inf \{ \alpha(S_\varepsilon) : \varepsilon > 0 \} = 0 .$$

Theorem 2.1. Let C be a convex, closed and bounded subset of a Banach space X with $\mathrm{int}(C) \neq \emptyset$. Then C has the drop property iff it is weakly compact (which means that the space is reflexive) and every support point is a point of continuity.

Proof. In view of Proposition 2.1 and Proposition 2.2 it remains to show the sufficiency. The generalized Drop theorem of Daneš [2] states that if $\inf \{ \|x - y\| : x \in A , y \in C \} > 0$ then there is an $x_0 \in A$ such that $D(x_0, C) \cap A = \{ x_0 \}$.

Assume that C fails to have the drop property, i.e. there is a closed set A, $A \cap C = \emptyset$ so that for each $x \in A$, $D(x,C) \cap A \neq \{x\}$. Using the theorem of Daneš we may construct inductively a sequence of distinct points $\{x_n\} \subset A$, $x_{n+1} \in D(x_n, C)$ and a sequence $\{y_n\} \subset C$ with $\| x_n - y_n \| < 1/n$, $n = 1,2,\dots$ Since C is weakly compact, choose a weakly convergent subsequence $\{ y_{n_k} \}$ and let $y \in C$ be its weak limit. Then,

$$y \in cl_w(\{x_{n_k}\}) \subset cl_w(co(\{x_n\})) = cl(co(\{x_n\})) .$$

As in Lemma 1 and Lemma 2 [5] we may prove that $co(\{x_n\}) \cap C = \emptyset$. Therefore, y is a boundary point of C. Since $int(C) \neq \emptyset$ then it follows from Corollary 2 ([3], p.64) that y is a support point of C . Hence, y is a point of continuity which implies that $y_{n_k} \to y$ and consequently, $x_{n_k} \to y$. Since A is closed, $y \in A$, i.e. $A \cap C \neq \emptyset$. This contradiction completes the proof.

Using the theorem of Daneš it is easy to observe that every convex and compact subset of a Banach space has the drop property.

Theorem 2.2. Let X be a reflexive Banach space and C be a convex, closed and bounded subset of X with the drop property. If C is symmetric ($C = -C$) then C is either compact or has non-empty interior.

Proof. Assume that C is not compact. We shall prove that C is absorbing and putting this together with the fact that C is closed and convex subset of a Banach space, we shall deduce that $int(C) \neq \emptyset$. Suppose the contrary, i.e. there exists an element $x_0 \in X$ so that for each $t > 0$ we have $tx_0 \notin C$. Since C is not compact we may choose an $\varepsilon > 0$ such that C does not possess a finite 2ε - net in X . Thus, select inductively a sequence $\{e_i\}_{i=1}^{\infty} \subset C$ so that

$$\inf\{\|e_n - z\| : z \in lin(x_0, e_1, e_2, \dots, e_{n-1})\} \geq \varepsilon .$$

Since $tx_0 \notin C$ for $t > 0$, we may choose $y_1 \in \{\pm e_1\} \subset C$ such that for $x_1 = (x_0 + y_1)/2$, the assertion $tx_1 \in C$ fails for every $t > 0$. Indeed, assuming that for some $t_1, t_2 > 0$ we have both $d_1 = t_1(x_0 + e_1)/2 \in C$ and $d_2 = t_2(x_0 - e_1)/2 \in C$, then by the convexity of C ,

$$t_2/(t_1+t_2)\, d_1 + t_1/(t_1+t_2)\, d_2 = t_1 t_2/(t_1+t_2)\, x_0 \in C ,$$

which is impossible.

Analogously, having chosen $x_1, x_2, \dots, x_n$ and $y_1, y_2, \dots, y_n$, select $y_{n+1} \in \{\pm e_{n+1}\} \subset C$ so that for $x_{n+1} = (x_n + y_{n+1})/2$, the assertion $tx_{n+1} \in C$ fails for every $t > 0$.

Denote $A = \{x_n\}_{n=1}^{\infty}$. Evidently, $C \cap A = \emptyset$. Moreover, A is closed since for $n > m$

$$\|x_n - x_m\| = \|y_n/2 + (\lambda_0 x_0 + \sum_{i=1}^{n-1} \lambda_i y_i)\| \geq \varepsilon/2 .$$

On the other hand we have by the construction that $x_{n+1} \in D(x_n, C)$, which is in contradiction with the fact that C has the drop property. This concludes the proof.

Theorem 2.3. Let X be a non-reflexive Banach space and C be a convex, closed and bounded subset of X . Then C has the drop property iff C is compact.

Proof. Let C have the drop property. Without loss of generality we may assume that $0 \in C$. Suppose that for every $x \in X$ there exists a convex combination

$$y = \lambda_o x + \sum_{i=1}^{n} \lambda_i y_i \in C ,$$

where $\lambda_o > 0$, $y_i \in C$.

It follows from $0 \in C$ that $z = \sum_{i=1}^{n} \lambda_i y_i \in C$ and hence, $\lambda_o x = y - z$, i.e. the set $K = C - C$ is absorbing. Moreover, K is convex by the convexity of C . It follows from Proposition 2.3 that K is weakly compact and in particular K is closed. Thus, $\operatorname{int}(K) \neq \emptyset$ which together with the weak compactness of K gives reflexivity of the space. Therefore, there exists a point $x_o \in X$ such that for every convex combination

$$\lambda_o x_o + \sum_{i=1}^{n} \lambda_i y_i \notin C ,$$

where $\lambda_o > 0$, $y_i \in C$.

Then, proceeding as in the proof of Theorem 2.2 we may show that the assumption C is not compact implies that C does not possess the drop property and this completes the proof.

Remark. We may ask if the assertion of Theorem 2.2 is valid without the assumption that the set C is symmetric.

Example. Let $X = l_2$ and C be the unit ball of l_1 considered as a subset of l_2 . Then C is closed, convex and bounded, every support point of C is a point of continuity but according to Theorem 2.2 C does not have the drop property.

References

1. J. Danes. A geometric theorem useful in nonlinear functional analysis. Boll. Un. Mat. Ital., 6, 1972, 369 - 375.
2. J. Danes. Equivalence of some geometric and related results of nonlinear functional analysis. Comment. Math. Univ. Carolinae, 26, 3, 1985, 443 - 454.
3. R. Holmes. Geometric Functional Analysis and its Applications. Springer, Berlin - Heidelberge - New York, 1975.
4. D. N. Kutzarova. A sufficient condition for the drop property. Compt. rend. Acad. bulg. sci., 39, 7, 1986, 17 - 19.
5. V. Montesinos. Drop property equals reflexivity. Studia Math. (to appear).
6. J. P. Penot. The Drop theorem, the Petal theorem and Ekeland´s variational principle. Nonlinear Anal. T. M. & Appl. (to appear).
7. S. Rolewicz. On drop property. Studia Math., 85, 1, 1987, 27 - 35.

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