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INTERPOLATING POLYNOMIALS ON THE TRIANGLE

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1. Introduction. Consider the mesh of points

$$(i/n,\ 1 - j/n), \qquad 0 \le i \le j \le n, \tag{1}$$

on the standard triangle with vertices at (0,0), (1,0) and (0,1). Given a function $f(x,y)$ defined on the standard triangle, there exists a unique polynomial $P_n(x,y)$ of degree at most n in x and y which interpolates f at the $\frac{1}{2}(n+1)(n+2)$ points of the above mesh. See, for example, Mitchell and Phillips [2]. The polynomial P_n can be written in a Lagrange form and can be computed very efficiently by an iterative process which is similar to the Neville-Aitken algorithm for evaluating the interpolating polynomial in one variable. See Lee and Phillips [1]. These results can be recovered as a special case of the generalization which we describe in more detail in the next section.

2. The 'geometric' mesh. In his most interesting paper [3], which contains several historical insights, Schoenberg discusses Stirling's derivation [4] of the one-dimensional interpolating polynomial based on a set of points in geometric progression. In order to extend this to the triangle, we introduce a real parameter q, $0 < q < 1$, and define the q-integers

$$[r] = (1 - q^r)/(1 - q), \qquad r = 0,1,\dots . \tag{2}$$

For a fixed value of q, $0 < q < 1$, we define the triangular mesh of points

$$([i]/[n],\ 1 - [j]/[n]), \qquad 0 \le i \le j \le n. \tag{3}$$

The interpolating polynomial for f on this point set is

$$P_n(x,y) = \sum_{i,j} \alpha^n_{i,j}(x,y) f_{i,j}, \tag{4}$$

where

$$f_{i,j} = f([i]/[n],\ 1 - [j]/[n]) \tag{5}$$

and

$$\alpha^n_{i,j}(x,y) = K^n_{i,j} \prod_{\nu=0}^{i-1} \left(x - \frac{[\nu]}{[n]}\right) \prod_{\nu=j+1}^{n} \left(y - 1 + \frac{[\nu]}{[n]}\right) \prod_{\nu=0}^{j-i-1} \left(1 - q^\nu x - y - \frac{[\nu]}{[n]}\right), \quad (6)$$

with

$$K^n_{i,j} = \frac{q^{\frac{1}{2}j(j+1-2n)}\ [n]^n}{[i]!\ [j-i]!\ [n-j]!}\ . \quad (7)$$

In (4) the sum is over $0 \leq i \leq j \leq n$ and in (7) the generalized factorial is defined as

$$[i]! = [i].[i-1]\ \dots\ [1]$$

for any positive integer i and $[0]! = 1$. The Lagrange coefficient (6) satisfies

$$\alpha^n_{i,j}\ ([s]/[n],\ 1 - [t]/[n]) = \delta_{is}.\delta_{jt},$$

for $0 \leq s \leq t \leq n$ and $0 \leq i \leq j \leq n$, where δ_{is} is the Kronecker delta.

The following process for evaluating $P_n(x,y)$ is much more efficient than merely using (4)-(7). For any point (x,y), let

$$f^o_{i,j} = f_{i,j},$$

as in (5), and define $f^m_{i,j}$ recursively from

$$q^j \frac{[m]}{[n]} f^m_{i,j} = \left(y - 1 + \frac{[j+m]}{[n]}\right) f^{m-1}_{i,j} + \left(1 - q^{j-i}x - y - \frac{[j-i]}{[n]}\right) f^{m-1}_{i,j+1}$$

$$+ q^{j-i}\left(x - \frac{[i]}{[n]}\right) f^{m-1}_{i+1,j+1} \quad (8)$$

for $0 \leq i \leq j \leq n-m$. We compute the $f^m_{i,j}$ for $m = 0,1,\dots,n$. We call this the Neville -Aitken algorithm since it is analogous to the well known recursive process for computing the interpolating polynomial in one variable. An induction argument shows that $f^m_{i,j}(x,y)$ is the interpolating polynomial for $f(x,y)$ on the triangular mesh of points

$$([i+s]/[n],\ 1 - [j+t]/[n]), \qquad 0 \leq s \leq t \leq m.$$

In particular,

$$f^n_{0,0}(x,y) = P_n(x,y). \quad (9)$$

If we take the limit as $q \to 1$, the q-integer $[r]$, as defined by (2), tends to the ordinary integer r. We then obtain special cases of (5)-(9) for the interpolating polynomial (4) on the mesh of points (1).

3. The three-pencil mesh. We observe that the points of the 'geometric' mesh (3) are points of intersection of the three systems of lines

$$x = [\nu]/[n], \qquad 0 \le \nu \le n, \tag{10a}$$

$$y = 1 - [\nu]/[n], \qquad 0 \le \nu \le n, \tag{10b}$$

$$q^{\nu}x + y = 1 - [\nu]/[n], \qquad 0 \le \nu \le n. \tag{10c}$$

Each point $([i]/[n],\ 1 - [j]/[n])$ is the point of intersection of three lines, one from each system. (We choose the line (10a) with $\nu = i$, (10b) with $\nu = j$ and (10c) with $\nu = j-i$.) Clearly the systems (10a) and (10b) are sets of parallel lines, while (10c) is a pencil of concurrent lines, the point of concurrence (or vertex) being

$$\left(\frac{1}{1-q^n},\ \frac{-q^n}{1-q^n}\right). \tag{11}$$

Thus in (10) we have three pencils of lines, two pencils having vertices at infinity and the third having the finite vertex (11). As $q \to 1$ the vertex (11) also goes off to infinity and (10) becomes three systems of parallel lines, corresponding to the mesh (1).

This observation prompts the question: can we find a triangular mesh based on three pencils of lines, each pencil having a finite vertex? An affirmative answer is provided by studying the set of points

$$P_{i,j} = \left(\frac{(1-a[n])q^{n-j}}{(1-aq^{n-j}[i]-b[n-j])}\cdot\frac{[i]}{[n]},\ \frac{(1-b[n])q^{-j}}{(1-aq^{n-j}[i]-b[n-j])}\left(1-\frac{[j]}{[n]}\right)\right), \tag{12}$$

for $0 \le i \le j \le n$. This mesh depends on the three parameters a, b and q. The 'geometric' mesh (3) can be recovered as a special case from (12) by taking $a = 0$, $b = 1-q$. Each point $P_{i,j}$ is the point of intersection of three lines, one from each of the systems

$$\left(\frac{1-a[\nu]}{1-a[n]}\right)x + [\nu]\left(\frac{1-q-b}{1-b[n]}\right)y = \frac{[\nu]}{[n]}, \qquad 0 \le \nu \le n, \tag{13a}$$

$$-\left(\frac{a[\nu]}{1-a[n]}\right)x + \left(\frac{1-b[\nu]}{1-b[n]}\right)y = \frac{[\nu]}{[n]}, \qquad 0 \le \nu \le n, \tag{13b}$$

$$\left(\frac{1-a[\nu]}{1-a[n]}\right)x + \left(\frac{1-b[\nu]}{1-b[n]}\right)y = \frac{[\nu]}{[n]}, \qquad 0 \le \nu \le n. \tag{13c}$$

(We need to choose $\nu = i$, $n-j$ and $n-j+i$ in (13a), (13b) and (13c) respectively.) Each of the above systems is, in fact, a pencil of lines. The vertices of the three pencils are respectively

$$\left(0, \frac{1-a[n]}{(1-q-b)[n]}\right), \left(1 - \frac{1}{a[n]}, 0\right), \left(\frac{1-a[n]}{(b-a)[n]}, \frac{-(1-b[n])}{(b-a)[n]}\right). \qquad (14)$$

As well as the 'geometric mesh', another interesting special case is obtained by taking $a = b$, when we find that two vertices are finite and the third is at infinity.

If we write the vertices (14) in the form

$$(0,\alpha), \quad (\beta,0), \quad (\gamma,1-\gamma), \qquad (15)$$

we can obtain a, b and q in terms of α, β and γ. Thus, for a given positive integer n, we can construct a three-pencil mesh based on any configuration of vertices (15).

For the general three-pencil mesh (12) we can easily write down a Lagrange form of the interpolating polynomial. Note how readily we obtain (6), knowing (10). In the three-parameter case the key step is to substitute (13) for (10). Likewise we can extend the Neville-Aitken algorithm to the three-parameter case.

References

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