CONSTRUCTIVE THEORY OF FUNCTIONS'87

SOFIA, 1988

SOME MORE UNIFORMLY CONVEX SPACES

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1. Introduction. The idea of a uniformly convex Banach space was first introduced by Clarkson[1] in 1936. There then followed a period of intense investigation into the properties of such spaces and the identification of examples of these spaces. We shall discuss a little of the history of this development, but our main concern will be to provide some new examples of constructions which lead to uniformly convex spaces. These constructions arise in a natural way from considerations of approximation theory, as we shall indicate.

DEFINITION. *(Clarkson 1936) A Banach space X is uniformly convex if to each ϵ, $0 < \epsilon \le 2$ there corresponds a $\delta(\epsilon) > 0$ such that the conditions*

$$\|x\| = \|y\| = 1, \|x-y\| \ge \epsilon, \quad x, y \in X$$

imply

$$\|\frac{1}{2}(x+y)\| \le 1 - \delta(\epsilon).$$

We now list some facts about uniformly convex spaces, which were discovered in a short period following the introduction of the idea by Clarkson.

(i) $L_p[0,1]$ is uniformly convex $1 < p < \infty$. So are Euclidean spaces of all dimensions, Hilbert spaces and Hyper-Hilbert spaces (Hilbert spaces which were not separable!).

(ii) X is uniformly convex implies X is reflexive (Milman- Pettis, 1938-1939)

(iii) Every bounded sequence in a uniformly convex space X admits a subsequence whose arithmetic means are norm convergent (Kakutani, 1938). This can be rephrased as X is uniformly convex implies that X has the Banach-Saks property.

(iv) If $\sum_{n=1}^{\infty} x_n$ is an unconditionally convergent series in a uniformly convex Banach space X, then $\sum_{n=1}^{\infty} \delta(\|x_n\|) < \infty$, where

$$\delta(\epsilon) = \inf\{1 - \|\frac{1}{2}(x+y)\| : \|x\| = \|y\| = 1, \|x-y\| = \epsilon\}.$$

When presented with such a list of results one could perhaps conclude that Clarkson introduced a definition which he thought would have intrinsic interest. However, there was a definite purpose to the motivation behind the idea. There are two results from complex function theory that were well known to workers in Banach spaces at the time of Clarkson's definition. They are as follows. (a) If f is a complex-valued function of bounded variation on [0,1], then the derivative of f exists almost everywhere with respect to Lebesgue measure.
(b) Also, if f is an absolutely continuous, complex-valued function on [0,1] then

$$f(t) = f(0) + \int_0^t f'(s)ds.$$

What interested researchers in the field at the time was that the above results cannot be generalised easily to the case where the range of f is a Banach space rather than the complex numbers. Bochner[2] had shown that his integral possessed flaws in the sense that neither of the above results continued to hold when the range of f is allowed to be an arbitrary Banach space.

So Clarkson, at the suggestion of Tamarkin it seems, posed the question what conditions on X are needed for X-valued functions to reproduce the above results for complex-valued functions? In his paper Clarkson was completely successful in showing that if X is uniformly convex then the results for complex valued functions continue to hold.

There followed some fifteen years later a paper by McShane[3,1950], who looked at $L_p(S,X)$ which is the space of all Bochner integrable functions from the measure space S to the Banach space X for which

$$|||f||| = \left\{ \int_S ||f(s)||^p ds \right\}^{\frac{1}{p}}$$

is finite. His result was that $L_p(S,X)$ is uniformly convex if and only if $L_p(S)$ and X are uniformly convex.

<u>2. More Uniformly Convex Function Spaces.</u> These arise quite naturally in some approximation - theoretic generalisations although as early as 1940 one can see evidence that people like Day[4] were thinking in similar directions.

THEOREM. *Suppose G is a proximinal subspace of $C(T)$ possessing a continuous proximity map $A : C(T) \to G$. Then $U = \overline{C(S) \otimes G}$ is a proximinal subspace of $C(S \times T)$.*

PROOF: Take $f \in C(S \times T)$. For each fixed s in S, $f_s \in C(T)$ where f_s is the s-section of f, so that $f_s(t) = f(s,t)$. Thus we can take Af_s for each $s \in S$. By the continuity of A, Af_s is in $C(S)$ and so $u_0(s,t) = Af_s(t)$ is in $\overline{C(S) \otimes G}$. Then for each $s \in S$,

$$\begin{aligned} & ||f_s - Af_s||_\infty \le ||f_s - g||_\infty \quad \text{for all } g \in G \\ \Rightarrow \quad & \sup_{s \in S} ||f_s - Af_s||_\infty \le \sup_{s \in S} ||f_s - g||_\infty \\ \Rightarrow \quad & || ||f_s - Af_s||_T ||_S \le || ||f_s - u_s||_T ||_S \\ \text{so that} \quad & ||f - u_0||_\infty \le ||f - u||_\infty \quad \text{for all } u \in C(S) \otimes G. \end{aligned}$$

Thus u_0 is a best approximation to f from $U = \overline{C(S) \otimes G}$.

This result can be reinterpreted as follows, by recalling that $C(S) \otimes X$ is isometrically isomorphic to $C(S, X)$ whenever S is compact Hausdorff and X is a Banach space. Also, $C(S \times T)$ is isometrically isomorphic (under the obvious isometry which utilises sections) to $C(S, C(T))$.

THEOREM. *If G is a proximinal subspace of the Banach space X and there is a continuous proximity map from X onto G, then $C(S, G)$ is proximinal in $C(S, X)$.*

A glance at the previous proof shows us that we can endow $C(S, X)$ with a norm other than $\|.\|_\infty$. The essential property is that of monotonicity.

DEFINITION. *A norm on $C(S)$ is called monotone if the inequalities $0 \leq x \leq y$ imply $\|x\| \leq \|y\|$.*
A norm on C(S) is called lattice if the inequality $|x| \leq |y|$ implies $\|x\| \leq \|y\|$.

Of course it is immediate from the definition that a lattice norm is a monotone norm.

Suppose α is a monotone norm on $C(S)$. Then there is a natural way in which α can be "lifted" to $C(S, X)$ where X is an arbitrary Banach space. For any $f \in C(S, X)$ we simply define the norm of f by the equation

$$|||f||| = \alpha(Jf) \qquad \text{where} \qquad (Jf)(s) = \|f(s)\|.$$

Sometimes we abuse notation by writing

$$|||f||| = \alpha(\|f(s)\|).$$

Then we can recover the above reformulated result in greater generality.

THEOREM. *Let G be a proximinal subspace of the Banach space X possessing a continuous proximity map from X onto G. Let α be a monotone norm on $C(S)$. Then $C(S, G)$ is α-proximinal in $C(S, X)$.*

However, we are interested in the geometry of $C(S, X)$ when a monotone norm α is imposed on this linear space. The following was already known.

THEOREM. *The space $C(S, X)$ is strictly convex whenever α is a strictly convex, monotone norm on $C(S)$ and X is a strictly convex normed linear space.*
In addition, if α is a lattice norm, then $C(S, X)$ is strictly convex if and only if $[C(S), \alpha]$ and X are both strictly convex.

These results are all contained in a preprint by Cheney, von Golitschek and Light[5]. Our main purpose is to point out that the following improvement of this result holds.

THEOREM. *The space $C(S, X)$ is uniformly convex whenever α is a uniformly convex, monotone norm on $C(S)$ and X is a uniformly convex normed linear space.*

In addition, if α is a lattice norm, then $C(S,X)$ is uniformly convex if and only if $[C(S),\alpha]$ and X are both uniformly convex.

COROLLARY. *If S is a compact Hausdorff space, μ is a regular Borel measure on S, and $1 < p < \infty$, then $L_p(S,X)$ is uniformly convex if and only if X is uniformly convex.*

Unfortunately, this result does not quite fit with the generality of McShane's result. The reason is that we began with a continuous function space. Thus to recover from this a result about L_p we must invoke some sort of density of the continuous functions in L_p. This cannot be done without some sort of assumption on the underlying measure space. The required generality can only be attained if we start with something a little more abstract than $C(S)$. There are several alternative approaches. The following one encompasses both the continuous function case and that for measurable functions.

We take S a set, X a Banach space, and $F(S)$ a linear space of functions from S into the real line R. We will assume that on a norm α is defined on this linear space in such a way that $[F(S),\alpha]$ is in fact a Banach space. Then we take $F(S,X)$ to be a linear space of functions from S into X such that for each $f \in F(S,X)$ the function $Jf : S \to R$ defined by $(Jf)(s) = \|f(s)\|$ is in $F(S)$.
A norm on $F(S,X)$ is defined by the equation

$$|||f||| = \alpha(Jf)$$

Monotonicity of α is enough to ensure that $|||.|||$ really is a norm, and without some such assumption the triangle inequality will not generally hold.
We need two more assumptions:
(a) F(S) is a Banach lattice containing the identity function.
(b) If $S = S_1 \cup \ldots \cup S_n$ and each S_i is the inverse image of an open set O_i in R under some function $g_i \in F(S)$, then there exist functions $f_1, \ldots, f_n$ in $F(S)$ such that
(1) $0 \leq f_i \leq 1$
(2) $\sum_{i=1}^{n} f_i = 1$
(3) The support of each f_i lies within S_i.
The effect of these assumptions is to impose some structure on $F(S)$ which is somehow inherited by $F(S,X)$. It is easily seen that if S is a measure space and $F(S)$ is one of the L_p spaces, then the above assumptions are valid. One simply takes characteristic functions of measurable sets. (There is a minor technicality in the way that the mapping J operates in this case because of the equivalence classes and the slight difference between the classical notion of measurability for real-valued functions and that of measurability for functions with range lying in a Banach space.) Also the above scenario contains the continuous function case. This may be seen as follows. If S is a compact Hausdorff space and the S_i are open sets, then a well-known corollary to Urysohn's lemma guarantees the existence of functions f_i with the above properties. We can now recover a theorem which encompasses the result of McShane.

THEOREM. *Let α be a monotone norm on $F(S)$. The space $F(S,X)$ is uniformly convex whenever X and $[F(S),\alpha]$ are themselves uniformly convex. If α is a lattice norm then the* *uniform convexity of $F(S,X)$ is equivalent to the uniform convexity of the two spaces $[F(S),\alpha]$ and X.*

COROLLARY. *Let (S,Σ,μ) be a finite measure space and X a uniformly convex Banach space. Then for $1 < p < \infty$, $L_p(S,X)$ is uniformly convex.*

We can also observe that the above theorem encompasses the earlier result where S was a compact Hausdorff space. It is worth observing in conclusion that the author learnt of similar work by a group of Bulgarian mathematicians while attending the meeting which gave rise to these proceedings. Unfortunately, at the time of going to press references for these works and their exact content were unavailable to him. However, from the discussions it would seem that the two approaches are rather different in spirit.

References

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