CONSTRUCTIVE THEORY OF FUNCTIONS'87

SOFIA, 1988

ON SOME CONTRIBUTIONS OF HALÁSZ TO THE TURÁN POWER–SUM THEORY, II

Lee Lorch and Dennis Russell

1. Introduction.

For a fixed positive integer n, and complex numbers b_j, z_j $(j = 1, \ldots, n)$, the generalized power sums

$$g(\nu) := \sum_{j=1}^{n} b_j z_j^{\nu} \quad (\nu = 1,2,\ldots) \tag{1}$$

are studied in P. Turán's posthumously–published book [3]. In our previous paper [2], we provided some improvements to lower bounds for these sums which G. Halász had contributed to [3]. In both [2] and [3], the analysis depended on estimates of bounds for a specific set of polynomials π_k^* $(k = 2,3,\ldots)$ belonging to the class

$$P_k := \{p(\cdot) \mid p \in \pi_k,\ p(0) = 1,\ p(1) = 0,\ p(z) \neq 0\ (|z| < 1)\}$$

$$\|p\| := \sup_{|z|=1} |p(z)| = \sup_{\theta} |p(e^{i\theta})| . \tag{2}$$

Thus, for example, [3, Lemma 5.8] showed that for each $k \in \{2,3,\ldots\}$, there is a polynomial $\pi_k^* \in P_k$ such that

$$\rho_k := \|\pi_k^*\| < \exp(2/k) . \tag{3}$$

In fact, the choice of π_k^* (see [3, p.52]) actually gives

$$\rho_k = \exp\left\{-\frac{1}{\pi}\int_0^{\pi/2} \log\left[1 - \left[\frac{\sin(k+1)\beta}{(k+1)\sin\beta}\right]^2\right] d\beta\right\} \tag{4}$$

The improvements in [2] depended on refining (3) to $\rho_k < \exp\left\{\frac{3}{2(k+1)}\right\}$, by making more careful estimates of the integral in (4).

However, at the presentation of [2] to the Seminar on Approximation and Optimization (Havana, Cuba, January 1987), E.B. Saff kindly drew our attention to the paper of Lachance–Saff–Varga [1], which actually determines the extremal polynomials for P_k. They prove, in particular:

<u>Lemma 1</u> [1]. *For each* $k \in \mathbb{Z}_{++} := \{1,2,\ldots\}$, *there is a polynomial* $p_k^* \in P_k$ *such that*

$$\|p_k^*\| = \min_{p \in P_k} \|p\| = \left\{\cos \frac{\pi}{2(k+1)}\right\}^{-(k+1)} =: \mu_k . \tag{5}$$

Instead of proceeding with further refinements using (4), as announced in [2], we here apply Lemma 1 to obtain substantial improvements on [3, Theorem 7.1], which gives a lower bound for the generalized power sums of $g(\nu)$, and on [3, Theorem 19.2], which gives a lower bound for the solutions of an ordinary differential equation with constant coefficients.

As may easily be verified by differentiation and L'Hôpital's Rule,

$$2 = \mu_1 \geq \mu_k \downarrow 1 \text{ for } 1 \leq k \uparrow +\infty , \tag{6}$$

$$4 = \mu_1^2 \geq \mu_k^{k+1} \downarrow e^{\pi^2/8} = 3.4339\ldots \text{ for } 1 \leq k \uparrow +\infty , \tag{7}$$

where the monotonicity is strict in both cases.

<u>Lemma 2</u>. *The polynomial* p_k^* *in Lemma 1 satisfies*

$$\frac{1}{|p_k^*(z)|} \leq \mu_k^{2r/(1-r)} \text{ in } |z| \leq r \ (0 < r < 1) . \tag{8}$$

<u>Proof</u>. Follow the argument in [3, p.54]: if $H(z) := \log p_k^*(z)$ then $\operatorname{Re} H(z) = \log|p_k^*(z)| \leq \log \mu_k$ for $|z| = 1$, by Lemma 1. Then by the Hadamard–Carathéodory Theorem,

$$\max_{|z|=r} |H(z)| \leq \frac{2r}{1-r} \log \mu_k = \log[\mu_k^{r/(1-r)}] . \qquad \square$$

<u>Remark (a)</u>. One should compare (8) with the Corollary in [3, p. 54], which gives

$$\frac{1}{|\pi_k^*(z)|} \leq \exp\left\{\frac{4r}{k(1-r)}\right\} \text{ in } |z| \leq r \ (0 < r < 1) .$$

2. Generalized power sums.

With the definition of $g(\lambda)$ given in (1), and μ_k given by (5), we have the following result:

Theorem 1. *Let $m \in \mathbb{Z}_+$, $k \in \mathbb{Z}_{++}$. Then for $\min|z_j| = 1$, where the minimum is taken over $j = 1, \ldots, n$, we have*

$$\max|g(\nu)| \geq |g(0)|(kn+1)^{-1/2}\, 2^{-m-1}\, \mu_k^{-3n}, \tag{9}$$

where the maximum is taken over $\nu = m+1, \ldots, m+kn$.

Proof. This proof follows very closely that of [3, Theorem 7.1]; we give it in detail for the sake of completeness.

Define $F_1(z) := \prod_{j=1}^{n} \overset{*}{p}_k\left[\frac{z}{z_j}\right] =: \sum_{\nu=0}^{kn} d_\nu^{(1)} z^\nu$, $d_0^{(1)} = 1$.

Then

$$\begin{aligned}
\sum_{\nu=0}^{kn} |d_\nu^{(1)}| &\leq \sqrt{kn+1}\, \left[\sum_{\nu=0}^{kn} |d_\nu^{(1)}|^2\right]^{1/2} \\
&= \sqrt{kn+1}\left[\frac{1}{2\pi}\int_{-\pi}^{\pi} |F_1(e^{i\theta})|^2\, d\theta\right]^{1/2} \\
&\leq \sqrt{kn+1}\cdot \max_\theta |F_1(e^{i\theta})| \\
&\leq \sqrt{kn+1}\cdot \max_\theta |\overset{*}{p}_k(e^{i\theta})|^n = \sqrt{kn+1}\cdot \mu_k^n .
\end{aligned} \tag{10}$$

Denote $\frac{1}{F_1(z)} := \sum_{\nu=0}^{\infty} d_\nu^{(2)} z^\nu$. Then, by Lemma 2,

$$\max_{|z|=r} \frac{1}{|F_1(z)|} \leq \max_{|z|=r} \frac{1}{|\overset{*}{p}_k(z)|^n} \leq \mu_k^{2nr/(1-r)} \quad (0 < r < 1),$$

and so $|d_\nu^{(2)}| \leq \frac{1}{r^\nu}\mu_k^{2nr/(1-r)}$, which implies

$$\sum_{\nu=0}^{m} |d_\nu^{(2)}| \leq \frac{1}{r^m(1-r)}\, \mu_k^{2nr/(1-r)} .$$

The choice $r = \frac{1}{2}$ gives

$$\sum_{\nu=0}^{m} |d_\nu^{(2)}| \leq 2^{m+1}\, \mu_k^{2n} . \tag{11}$$

Write $s_m\left[\frac{1}{F_1},z\right] := \sum_{\nu=0}^{m} d_\nu^{(2)} z^\nu$ and denote

$$F_2(z) := 1 - F_1(z)\, s_m\left[\frac{1}{F_1},z\right] . \tag{12}$$

Then it follows by the argument in [3] (top of p. 75, using §6.2) that

$$F_2(z) = \sum_{\nu=m+1}^{m+kn} d_\nu^{(3)} z^\nu , \text{ where } \sum_{\nu=m+1}^{m+kn} d_\nu^{(3)} z_j^\nu = 1 \quad (j = 1, \ldots ,n) ,$$

so that

$$\sum_{\nu=m+1}^{m+kn} d_\nu^{(3)} g(\nu) = g(0) .$$

Consequently, by (12),

$$\max_{\nu=m+1,\ldots,m+kn} |g(\nu)| \ge |g(0)| \Big/ \sum_{\nu=m+1}^{m+kn} |d_\nu^{(3)}| \ge \frac{|g(0)|}{\sum_{\nu=0}^{kn} |d_\nu^{(1)}| \cdot \sum_{\nu=0}^{m} |d_\nu^{(2)}|} .$$

Using (10) and (11), we get, as required,

$$\max_{\nu=m+1,\ldots,m+kn} |g(\nu)| \ge |g(0)|(kn+1)^{-1/2}\, 2^{-m-1}\, \mu_k^{-3n} . \qquad \square$$

<u>Remark (b)</u>. By following the procedure of [3, §6.3], the inequality (9) remains true for any $m \in \mathbb{R}_+$, namely for $m + 1 \le \nu \le m + kn$ and ν an integer.

<u>Remark (c)</u>. Halász' inequality [3, Theorem 7.1] replaces the right of (9) by

$$|g(0)|(kn+1)^{-1/2}\, 2^{-m-1}\, e^{-6n/k} .$$

Since $\mu_k \le \rho_k \le e^{2/k}$, Theorem 1 above provides an improvement.

If, in Theorem 1, we choose $m = 0$, $k = n - 1$ (with $n \ge 2$) , then $\sqrt{kn+1} < n$ and $\mu_{n-1}^n \le \mu_1^2 = 4$, and we get the inequalities:

<u>Corollary 1.1</u>. *If* $n \in \{2,3, \ldots\}$ *and* $\min_{j=1,\ldots,n} |z_j| = 1$ *then*

$$\max_{\nu=1,2,\ldots,n(n-1)} |g(\nu)| \ge \frac{1}{2n} \cdot \mu_{n-1}^{-3n} |g(0)| \ge \frac{1}{128n} |g(0)| . \tag{13}$$

<u>Remark (d)</u>. This last result is considerably better than the Corollary in [3, p. 75], which states

$$\max_{\nu=1,2,\ldots,n^2} |g(\nu)| \ge \frac{1}{4e^6 n} |g(0)| .$$

Corollary 1.1 improves the constant by a ratio of $\frac{4e^6}{128} = 12.6\ldots$, as well as by bounding below the maximum over a substantially shorter ν–interval.

3. Application to ordinary differential equations.

Suppose that $y(t)$ is any solution of

$$y^{(n)}(t) + a_1 y^{(n-1)}(t) + \ldots + a_n y(t) = 0$$

with constant coefficients, expressed so that

$$(14) \qquad y(t) = \sum_{j=1}^{n} b_j e^{i\alpha_j t}, \quad \min_j \operatorname{Re}(\alpha_j) = 0 .$$

Theorem 2. *Let* $n,k \in \mathbb{Z}_{++}$, $a \geq 0$, $d > 0$, *and let* $y(t)$ *be given by* (14). *Then*

$$(15) \qquad \max_{a \leq t \leq a+d} |y(t)| \geq \tfrac{1}{2}(kn+1)^{-1/2}\, 2^{-ank/d}\, \mu_k^{-3n}\, |y(0)| .$$

Proof. (cf. [3, p. 216]). With b_j, a_j as in (14) above, let

$$m_1 := \frac{na}{d}, \; z_j := e^{id\alpha_j/n}, \; g(\nu) := \sum_{j=1}^{n} b_j z_j^{\nu} = \sum_{j=1}^{n} b_j e^{i\nu d\alpha_j/n} .$$

Now, by Theorem 1 and Remark (b),

$$\max_{m_1 \leq \nu \leq m_1+kn} |g(\nu)| \geq |g(0)|(kn+1)^{-1/2}\, 2^{-m_1-1}\, \mu_k^{-3n}$$

namely

$$\max_{a \leq \frac{\nu d}{n} \leq a+kd} \Big| \sum_{j=1}^{n} b_j e^{i\nu d\alpha_j/n} \Big| \geq \Big| \sum_{j=1}^{n} b_j \Big| \cdot \tfrac{1}{2}(kn+1)^{-1/2}\, 2^{-na/d}\, \mu_k^{-3n} .$$

Replacing d by $\frac{d}{k}$ and noting that $\max|y(t)|$ for $t \in [a,a+d]$ is not less than the maximum when t runs over the values $\frac{\nu d}{kn}$ in this interval, we obtain (15). □

Since the left side of (15) is independent of k, the choice of k in Theorem 2 is at our disposal. Denote

$$(16) \qquad \lambda_k := \frac{\log(\mu_k^{k+1})}{\log 2} = \frac{(k+1)^2 \log \sec \frac{\pi}{2(k+1)}}{\log 2} \quad (k = 1,2,\ldots) .$$

Then

$$(17) \qquad 2 = \lambda_1 \geq \lambda_k \downarrow \frac{\pi^2}{8 \log 2} = 1.7798\ldots \text{ for } 1 \leq k \uparrow +\infty .$$

Now $\mu_k^{-3n} = (\mu_k^{k+1})^{-3n/(k+1)} = 2^{-3\lambda_k n/(k+1)} \geq 2^{-3\lambda_{k_0} n/(k+1)}$ for $k \geq k_0 \geq 1$., and then

$$(2^{-ank/d})\,\mu_k^{-3n} \geq 2^{-n\left[\frac{ak}{d} + \frac{3\lambda_{k_0}}{k+1}\right]} \quad \text{for } k \geq k_0 .$$

If we choose $k := \left[\frac{\gamma}{2}\sqrt{\frac{d}{a}}\right]$ ($[\cdot]$ is here the greatest–integer function), then

$$\frac{ak}{d} + \frac{3\lambda_{k_0}}{k+1} \leq \left[\frac{1}{2}\gamma + \frac{3\lambda_{k_0}}{\frac{1}{2}\gamma}\right]\sqrt{\frac{d}{a}}$$

and we can minimize the right hand side by choosing $\gamma = \gamma_0 := 2\sqrt{3\lambda_{k_0}}$. Consequently, when $k := \left[\frac{\gamma_0}{2}\sqrt{\frac{d}{a}}\right]$ we have

$$2^{-ank/d}\,\mu_k^{-3n} \geq 2^{-\gamma_0 n\sqrt{a/d}}$$

Also, if $k \geq k_0$, $n \geq n_0$, then

$$kn + 1 \leq \left[1 + \frac{1}{k_0 n_0}\right] kn \leq \left[1 + \frac{1}{k_0 n_0}\right]\frac{1}{2}\gamma_0 n\sqrt{\frac{d}{a}} .$$

To ensure that $k \geq k_0$ it suffices to require that $\frac{1}{2}\gamma_0\sqrt{\frac{d}{a}} \geq k_0$, namely that

$$0 < a < \left[\frac{\gamma_0}{2k_0}\right]^2 d = \frac{3\lambda_{k_0}}{k_0^2}\, d .$$

Hence we obtain the following corollary of Theorem 2:

<u>Corollary 2.1</u>. *Let* λ_k *be as defined in* (16). *Let* $n_0, k_0 \in \mathbb{Z}_{++}$,

$$\gamma_0 := 2\sqrt{3\lambda_{k_0}},\ \kappa_0 := \left[2\left[1 + \frac{1}{k_0 n_0}\right]\right]^{-1/2},\ 0 < a < \frac{3\lambda_{k_0}}{k_0^2}\, d .$$

Then, for $n \geq n_0$ *and* $y(t)$ *as defined in* (14), *we have*

$$\text{(18)} \qquad \max_{a \leq t \leq a+d} |y(t)| \geq |y(0)|\, \kappa_0 (\gamma_0 n\sqrt{d/a})^{-1/2}\, 2^{-\gamma_0 n\sqrt{a/d}} .$$

With the choice $d = 1$, $a = n^{-2}$, $n \geq n_0 := 2$, we have $\frac{a}{d} = \frac{1}{n^2} \leq \frac{1}{4}$, and we may therefore choose $k_0 = 4$, for which $\frac{3\lambda_{k_0}}{k_0^2} = 0.33936\ldots > \frac{1}{4}$, and then $\gamma_0 = 2\sqrt{3\lambda_{k_0}} = 4.66037\ldots$ and $\kappa_0 = \left[2\left[1 + \frac{1}{k_0 n_0}\right]\right]^{-1/2} = \frac{2}{3}$. Substituting in Corollary 2.1, we obtain

Corollary 2.2. *For* $n \geq 2$ *and* $y(t)$ *as in* (14), *we have*

(19)
$$\max_{n^{-2} \leq t \leq 1+n^{-2}} |y(t)| \geq \frac{1}{82} n^{-1} |y(0)| .$$

Remark (e). Compare the Corollary in [3, p. 217], which has $\frac{1}{5e^6} n^{-1} |y(0)|$ on the right of the inequality. Corollary 2.2 therefore improves the constant by a ratio of $\frac{5e^6}{82} = 24.6$.

For an optimization of Corollary 2.1, let

$$d = 1, \; a = (\delta n)^{-2} , \; n \geq n_0 \geq 1 .$$

Then $(\gamma_0 n \sqrt{d/a}\,)^{-1/2} 2^{-\gamma_0 n \sqrt{a/d}} = (\gamma_0 n^2 \delta)^{-1/2} 2^{-\gamma_0 \delta}$.

$$= n^{-1} \gamma_0^{-1/2} \cdot x^{1/2} e^{-(\gamma_0 \log 2) x} , \; x = 1/\delta .$$

Now, $x^{1/2} e^{-(\gamma_0 \log 2) x}$ achieves its unique maximum value when $x = 1/\delta = 1/(2\gamma_0 \log 2)$, namely when $\delta = 2\gamma_0 \log 2$; and with this choice of δ we easily obtain

(20)
$$(\gamma_0 n \sqrt{d/a})^{-1/2} 2^{-\gamma_0 n \sqrt{a/d}} = \frac{1}{\gamma_0 n \sqrt{2e \log 2}} .$$

For the choice of k_0 , we first have $\sqrt{\frac{a}{d}} = \frac{1}{\delta n} \leq \frac{1}{2 n_0 \gamma_0 \log 2}$ $(n \geq n_0)$; but we require also, to apply Corollary 2.1, that $\sqrt{\frac{a}{d}} < \frac{\gamma_0}{2k_0}$, and so it suffices to choose k_0 so that we have

$$k_0 < n_0 \gamma_0^2 \log 2 = 12 n_0 \lambda_{k_0} \log 2 .$$

Since $\lim_{k \to \infty} \lambda_k = \pi^2/(8 \log 2)$ and $\lambda_k \downarrow$, by (17), we may, a fortiori, choose k_0 so that

$$k_0 \leq 12 n_0 \log 2 \cdot \pi^2/(8 \log 2) = \frac{3}{2} n_0 \pi^2 .$$

As in Corollary 2.2, take $n_0 = 2$; then we want $k_0 \leq 3\pi^2 = 29.608. \ldots$ Hence we choose $k_0 = 29$. Then

$$\kappa_0 = \left[2\left[1 + \frac{1}{k_0 n_0} \right] \right]^{-1/2} = \sqrt{\frac{29}{59}} \; , \; \gamma_0 = \sqrt{12 \lambda_{k_0}} = 4.62255. \ldots$$

The coefficient of $|y(0)|$ on the right of (18) then becomes, from (20),

$$\frac{\kappa_0}{n \gamma_0 \sqrt{2e \log 2}} = 0.0781296 \ldots n^{-1} > \frac{5}{64n} .$$

Thus we have proved:

Corollary 2.3. *Let* $n \geq 2$ *and* $y(t)$ *be as in* (14). *Let* $\delta := 4\sqrt{3\lambda_{29}} \log 2 = 6.40822\ldots$ *Then*

$$\max_{(\delta n)^{-2} \leq t \leq 1+(\delta n)^{-2}} |y(t)| \geq \frac{5}{64} n^{-1} |y(0)| .$$

4. **Acknowledgment**. We acknowledge support of our work by the Natural Sciences and Engineering Research Council of Canada.

References

[1] M. Lachance, E.B. Saff and R.S. Varga: Inequalities for polynomials with a prescribed zero, Math. Z. 168 (1979) 105–116.

[2] L. Lorch and D.C. Russell: On some contributions of Halász to the Turán power–sum theory, Proceedings Int. Sem. Approximation and Optimization (Havana, Cuba, January 1987); [to appear].

[3] P. Turán: On a New Method of Analysis and its Applications, John Wiley & Sons, New York, 1984.

Lee Lorch and Dennis Russell
Department of Mathematics
York University
North York, Ontario
Canada M3J 1P3