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A SIMPLE PROOF OF AN INEQUALITY IN THE THEORY OF n-WIDTHS

Yuly Makovoz

The following theorem plays a key role in the theory of n-widths of Sobolev classes.

THEOREM. *For any natural n, m, $n < m$, there exists a subspace $\Gamma \subset \mathbb{R}^m$, $\dim \Gamma \geq m - n$, such that for $x \in \Gamma$*

$$\|x\|_2 \leq Cn^{-1/2}(\log \frac{em}{n})^{1/2} \cdot \|x\|_1 \ , \tag{1}$$

where C is an absolute constant.

(Here and below, $x = (\xi_1, \cdots, \xi_m)$, $\|x\|_p = (|\xi_1|^p + \cdots + |\xi_m|^p)^{1/p}$, $1 \leq p < \infty$, $\|x\|_\infty = \max |\xi_i|$.) The original proof of this theorem given by Kashin [1] (with the power of the logarithmic factor equal to 3/2) was rather complicated. Developing Kashin's method, Garnaev and Gluskin [2] obtained a simpler proof; they also showed that, up to a constant factor, the estimate (1) is the best possible. The proof in [2] is based on a deep fact from the geometry of the unit sphere (the isoperimetric inequality) and on a geometric argument from the paper [3]. Our proof below follows the general idea of [2], but is quite elementary and self-contained.

Let $S = S^{m-1} = \{x \in \mathbb{R}^m : \|x\|_2 = 1\}$ and let μ be the normalized Lebesgue measure on S, $\mu(S) = 1$. For $x \in \mathbb{R}^m$ and $y_1, \ldots, y_n \in S$, let

$$F(x, \overline{y}) = n^{-1}[|(x, y_1)| + \cdots + |(x, y_n)|] \ .$$

Denote P the measure on the product set $Y = (S^{m-1})^n$ of all multivectors $\overline{y} = (y_1, \ldots, y_n)$, $y_1, \ldots, y_n \in S$, defined by $dP(\overline{y}) = d\mu(y_1) \cdot \ldots \cdot d\mu(y_n)$.

LEMMA. *For any $x \in S$,*

$$P(\overline{y} : (.01)m^{-1/2} \leq F(x, \overline{y}) \leq 3m^{-1/2}) > 1 - \exp(-n/2) \tag{2}$$

Proof. Let

$$E_n(t) = \int_Y \exp\big(tF(x,\overline{y})\big)dP(\overline{y})$$

Obviously, $E_n(t)$ does not depend on x. If $t > 0$, $b > 0$, then $E_n(t) \geq \exp(bt) \cdot P(\overline{y} : F(x,\overline{y}) > b)$, hence

(3) $$P(\overline{y} : F(x,\overline{y}) > b) \leq E_n(t)\exp(-bt)$$

Similarly, for any $t < 0$, $a > 0$,

(4) $$P(\overline{y} : F(x,\overline{y}) < a) \leq E_n(t)\exp(-at)$$

Taking $x = (1,0,\ldots,0)$, $y = (\eta_1,\ldots,\eta_m) \in S$, we have

(5) $$E_1(t) = \int_S \exp(t|(x,y)|)\, d\mu(y) = \int_S \exp(t|\eta_1|)\, d\mu$$

For $0 \leq \alpha < \beta \leq 1$, the measure of the strip $\{y \in S : \alpha \leq \eta_1 \leq \beta\}$ is proportional to

$$\int_{\sqrt{1-\beta^2}}^{\sqrt{1-\alpha^2}} (1-r^2)^{-1/2} \cdot \big(\frac{d}{dr}V_{m-1}(r)\big)\, dr \ ,$$

where $V_{m-1}(r)$ is the $(m-1)$-dimensional volume of the ball $\eta_2^2 + \cdots + \eta_m^2 \leq r^2$. We have $V_{m-1}(r) = r^{m-1}V_{m-1}(1)$. Hence

$$\mu\{y \in S : \alpha \leq \eta_1 \leq \beta\} = (2I_m)^{-1} \int_{\sqrt{1-\beta^2}}^{\sqrt{1-\alpha^2}} (1-r^2)^{-1/2} r^{m-2}\, dr$$
$$= (2I_m)^{-1} \int_\alpha^\beta (1-u^2)^{\frac{m-3}{2}}\, du \ ,$$

where I_m is the normalizing factor,

$$I_m = \int_0^1 (1-u^2)^{\frac{m-3}{2}}\, du > \int_0^{1/\sqrt{m}} (1-u^2)^{\frac{m-3}{2}}\, du > (1-1/m)^{\frac{m-1}{2}} \cdot m^{-1/2} > (em)^{-1/2} \ .$$

Returning to (5), we have

$$E_1(t) = I_m^{-1} \int_0^1 \exp(tu)(1-u^2)^{\frac{m-3}{2}}\, du$$

We now use the inequality $1 - u^2 \leq e^{-u^2}$ to get

$$E_1(t) < (em)^{1/2} \int_0^1 \exp(tu - mu^2/2 + 3u^2/2)\, du$$
$$< e^2 m^{1/2} \int_0^1 \exp(tu - mu^2/2)\, du \ .$$

Extending integration to $[0,\infty)$ and substituting $v = m^{1/2}(u - t/m)$, one has

(6) $$E_1(t) < \exp(2 + t^2/2m) \int_{-t/\sqrt{m}}^{\infty} \exp(-v^2/2)\, dv$$

Since obviously $E_n(t) = \left(E_1(t/n)\right)^n$, (6) implies

$$E_n(t) < \exp(2n + t^2/2mn)\left[\int_{-t/(n\sqrt{m})}^{\infty} \exp(-v^2/2\, dv\right]^n \tag{7}$$

We now use (3) and (7) with $b = 3m^{-1/2}$, $t = 3m^{1/2}n$ (the integral in (7) is less than $\sqrt{2\pi}$):

$$P(\overline{y} : F(x, \overline{y}) > 3m^{-1/2}) < (e^{-5/2}\sqrt{2\pi})^n \ . \tag{8}$$

If $c > 0$, $\int_c^\infty \exp(-v^2/2)\, dv < (1/c)\exp(-c^2/2)$. Using this we set $a = (.01)m^{-1/2}$, $t = -100m^{1/2}n$ and combine (4) and (7):

$$P(\overline{y} : F(x, \overline{y}) < (.01)m^{-1/2}) < (.01e^3)^n \tag{9}$$

The estimate (2) follows from (8) and (9). ∎

Proof of the theorem. Let $B_p^m = \{x \in \mathbb{R}^m : \|x\|_p \leq 1\}$ and for a fixed natural ℓ, $1 \leq \ell \leq m$, let $B^{m,\ell}$ be the set of all vectors from B_1^m with coordinates of the form k/ℓ, $k \in \mathbb{Z}$. The cardinality of $B^{m,\ell}$, is less than 2^ℓ times the number of non-negative integer solutions of the inequality $\ell_1 + \cdots + \ell_m \leq \ell$, *i.e.* $|B^{m,\ell}| < 2^\ell\binom{m+\ell}{\ell}$. We set now

$$\ell = [An/\log(em/n)] \tag{10}$$

If $A > 0$ is sufficiently small, there exists $\overline{y}^* \in Y$ such that for all $x \in B^{m,\ell}$

$$(.01)m^{-1/2}\|x\|_2 \leq F(x, \overline{y}^*) \leq 3m^{-1/2}\|x\|_2 \tag{11}$$

Indeed, measure P of all $\overline{y} \in Y$ for which this is not true does not exceed, due to (2), $|B^{m,\ell}| \cdot \exp(-n/2)$. Using the inequality $\binom{m}{\ell} \leq (\frac{em}{\ell})^\ell$, we have

$$|B^{m,\ell}| \cdot \exp(-n/2) < 2^\ell(2em/\ell)^\ell \cdot \exp(-n/2) \ ,$$

which is less than one if $A > 0$ in (10) is sufficiently small. This implies the existence of $\overline{y}^*$.

Let Γ be the subspace of dimension $\geq (m - n)$ defined by

$$\Gamma = \{x \in \mathbb{R}^m : F(x, \overline{y}^*) = 0\} \ .$$

To complete the proof we show that if $x \in \Gamma \cap B_1^m$, then

$$\|x\|_2 \leq 301\ell^{-1/2} \ . \tag{12}$$

To this end, take $x \in B_1^m$ and let x' be the nearest to x in $B^{m,\ell}$ in the ℓ_∞^m-norm; $x'' = x - x'$. Then $x'' \in B_1^m \cap (1/\ell)B_\infty^m : = \sqcap_{m,\ell} = \sqcap$. We claim that

$$\sqcap = \ \text{conv}\ (V) \ , \tag{13}$$

where $V = V_{m,\ell}$ is the set of all vectors having exactly ℓ coordinates equal to $\pm 1/\ell$ with all the other coordinates equal to zero. Indeed, $\sqcap$ is a convex polytope and is therefore the convex hull of its vertices (extreme points). If $z \in \sqcap\backslash V$, then by perturbing one or two coordinates of z one obtains $z_1, z_2 \in \sqcap$ such that $z_1 \neq z_2$ and $z = (z_1+z_2)/2$; if $z \in V$, this is impossible. Thus V is the set of the vertices of $\sqcap$ and (13) follows. As a consequence we have $\|x''\|_2 \leq 1/\sqrt{\ell}$ and by (11), since $V \subset B^{m,\ell}$, $F(x'', \overline{y}^*) \leq \max\{F(z, \overline{y}^*) : z \in V\} \leq 3m^{-1/2}\ell^{-1/2}$. If we now assume that $x \in B_1^m$ and $\|x\|_2 > 301\ell^{-1/2}$, then $\|x'\|_2 \geq \|x\|_2 - \|x''\|_2 > 300\ell^{-1/2}$ hence $F(x', \overline{y}^*) > (.01) \cdot 300m^{-1/2}\ell^{-1/2}$ and $F(x, \overline{y}^*) \geq F(x', \overline{y}^*) - F(x'', \overline{y}^*) > 0$ which means that $x \notin \Gamma$. ∎

References

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Department of Mathematics
University of Lowell
Lowell, MA 01854
U.S.A.