CONSTRUCTIVE THEORY OF FUNCTIONS'87

SOFIA, 1988

ON THE SHARPNESS OF ERROR BOUNDS FOR COMPOUND QUADRATURE RULES IN THE SPACE OF RIEMANN INTEGRABLE FUNCTIONS

H. Mevissen, R.J. Nessel, E. van Wickeren

For functions f, Riemann integrable on the interval [0,1], i.e., $f \in R[0,1]$, consider the compound quadrature formula ($b_k \neq 0$, $x_k \in [0,1]$ for $k = 1,\ldots,s$)

$$Q_n f := \frac{1}{n} \sum_{i=1}^{n} \sum_{k=1}^{s} b_k f\left(\frac{i-1+x_k}{n}\right) .$$

It is well-known (cf. [1, p. 21]) that $Q_n f$ converges to $Qf := \int_0^1 f(x)dx$ on R[0,1] if and only if

$$(1) \qquad \sum_{k=1}^{s} b_k = 1$$

which will be assumed in the following. Concerning estimates for the error $R_n f := |Q_n f - Qf|$, appropriate measures of smoothness are given for continuous functions f on [0,1], i.e., $f \in C[0,1]$, by the (ordinary) modulus of continuity

$$\omega_k(f,\delta) := \|\omega_k(f,\cdot,\delta)\|_\infty := \sup_{0\leq x\leq 1} \omega_k(f,x,\delta),$$

$$\omega_k(f,x,\delta) := \sup\{|\Delta_h^k f(t)| : t, t+kh \in U_\delta(x)\},$$

$$\Delta_h^k f(t) := \sum_{j=0}^{k} (-1)^{k-j}\binom{k}{j} f(t+jh), \quad U_\delta(x) := [x-\delta,x+\delta] \cap [0,1],$$

and for $f \in R[0,1]$ by the τ-modulus (cf. [5, p. 19])

$$\tau_k(f,\delta) := \int_0^1 \omega_k(f,x,\delta)dx .$$

Thus, if Q_n is exact for all polynomials of degree $k-1$, then for each $f \in R[0,1]$ (see [3], also [5, p. 74] and the literature cited there)

$$R_n f \leq C_k \tau_k(f,1/n) \leq C_k \omega_k(f,1/n) .$$

The purpose of this paper is to show that the first inequality is sharp and that error estimates in terms of ω_k are indeed worse than those in terms of τ_k.

To this end, let $\omega(\delta)$ be an abstract modulus of continuity, i.e., $\omega(\delta)$ is continuous, subadditive, and increasing for $\delta > 0$ satisfying

(2) $$\omega(\delta) = o(1), \quad \delta = o(\omega(\delta)) .$$

<u>Theorem 1:</u> There exists a counterexample $f_\omega \in R[0,1]$ such that

$$\tau_1(f_\omega,\delta) = O(\omega(\delta)), \quad R_n f_\omega \neq o(\omega(1/n)), \quad \omega_1(f_\omega,\delta) \neq o(1) .$$

Even if one restricts to functions $f \in C[0,1]$, i.e., $\omega_1(f,\delta) = o(1)$, there occurs an essential difference between the two moduli. Indeed, let α be a function with

(3) $$\alpha(\delta) = o(1), \quad \omega(\delta) = o(\alpha(\delta)) .$$

<u>Theorem 2:</u> There exists $f_{\omega,\alpha} \in C[0,1]$ with

$$\tau_1(f_{\omega,\alpha},\delta) = O(\omega(\delta)), \quad R_n f_{\omega,\alpha} \neq o(\omega(1/n)), \quad \omega_1(f_{\omega,\alpha},\delta) \neq o(\omega(\delta)/\alpha(\delta)) .$$

Proceeding to moduli of order $k \geq 2$ one obtains yet a stronger result.

<u>Theorem 3:</u> There exists $f_\omega \in C[0,1]$ with

$$\tau_k(f_\omega,\delta) = O(\omega(\delta^k)), \quad R_n f_\omega \neq o(\omega(n^{-k})), \quad \omega_k(f_\omega,\delta) \neq o(\delta^{-1}\tau_k(f_\omega,\delta)) .$$

These results follow as applications of the subsequent quantitative extension of the uniform boundedness principle. Let X be a Banach space with norm $\|\cdot\|_X$ and X^* be the space of non-negative, sublinear, bounded functionals T on X, i.e., for $f,g \in X$ and scalars α

$$0 \leq T(f+g) \leq Tf + Tg, \qquad T(\alpha f) = |\alpha| Tf ,$$

$$\sup\{Tf: f \in X, \|f\|_X \leq 1\} < \infty .$$

Let I be the index set {1} or {1,2}, in the first case indices will be omitted.

Theorem 4: Let $\{\varphi_n^i\}$ be a strictly decreasing nullsequence for each $i \in I$, let $\sigma(\delta) > 0$ and ω be subject to (2). Suppose that for U_δ, T_n^i, $V_n \in X^*$ there are elements $h_n^i \in X$ satisfying ($i \in I$)

(4) $\|h_n^i\|_X = O(1)$,

(5) $U_\delta h_n^i \leq M \min\{1, \sigma(\delta)/\varphi_n^i\}$ $\qquad (\delta > 0, n \in \mathbb{N})$,

(6) $T_n^i h_n^i \neq o(1)$,

(7) $V_n h_n^2 = O(1)$,

(8) $V_n h_j^i = o(\omega(\varphi_n^2))$ $\qquad (j \in \mathbb{N})$.

Then there exists a counterexample $f_\omega \in X$ with

(9) $U_\delta f_\omega = O(\omega(\sigma(\delta)))$,

(10) $T_n^i f_\omega \neq o(\omega(\varphi_n^i))$,

(11) $T_n^2 f_\omega \neq o(V_n f_\omega)$.

For a proof (even for, e.g., countable I) see [2,4,6] and the literature cited there. Note that the Landau symbol in (8) depends upon $i \in I$, $j \in \mathbb{N}$.

Proof of Theorem 1: Let X be the space of functions $f \in R[0,1]$ for which the set $\{x: f(x) \neq 0\}$ is countable. Then X is a Banach space under $\|\cdot\|_\infty$. Setting $I = \{1\}$,

$T_n = R_n$, $V_n = 0$, $U_\delta f = \tau_1(f,\delta)$, and

$$h_n(x) = \begin{cases} 1; & x = (i-1+x_k)/n,\ i = 1,\dots,n; k = 1,\dots,s \\ 0; & \text{else,} \end{cases}$$

one has $h_n \in X$ and $\|h_n\|_X = 1$, thus (4). Let $\{y_j\}_{j=0}^m \subset [0,1]$ be arbitrary. Then

$$\sum_{j=1}^m |h_n(y_j) - h_n(y_{j-1})| \le 2\,ns$$

so that h_n is of bounded variation on [0,1] with Var $h_n \le 2\,ns$. This gives (cf. [5, p. 24])

$$\tau_1(h_n,\delta) \le \delta \operatorname{Var} h_n \le 2s\,\delta n\ ,$$

hence (5) with $\sigma(\delta) = \delta$ and $\varphi_n = 1/n$. Moreover, $T_n h_n = 1$ in view of (1), thus (6). Now (9) and (10) deliver the result since the only continuous function in X is the one, identically zero on [0,1], and this of course cannot be the counterexample f_ω, delivered from Theorem 4

Proof of Theorem 2: Let X = C[0,1] with norm $\|\cdot\|_\infty$ and G be an infinitely differentiable function on the real axis with

$$0 \le G(t) \le 1, \qquad \int_{-1}^{1} G(t)\,dt < 1\ , \qquad G(t) = \begin{cases} 0; & |t| \ge 1 \\ 1; & t = 0\ . \end{cases}$$

Let z_j, $1 \le j \le q \le ns$, be the different nodes of the compound quadrature formula and $c_0 > 0$ be such that

$$\min\{|z_j - z_{j-1}| : 2 \le j \le q\} \ge c_0/n\ .$$

Let $\{m_n\}$ be a subsequence with (cf. (3))

(12) $\qquad m_n \ge \max\{2/c_0, s\}n\ ,$

(13) $\qquad \omega(1/m_n)/\alpha(1/m_n) \le \omega(1/n)\ .$

With $I = \{1,2\}$ set $T_n^1 f = R_n f$, $T_n^2 f = \omega_1(f,1/m_n)$, $V_n = 0$, $U_\delta f = \tau_1(f,\delta)$, and

$$(14)\qquad h_n^1(x) = h_n^2(x) = g_n(x) := \sum_{j=1}^{q} G(m_n(x-z_j)) .$$

If $x \in [0,1]$ is such that $G(m_n(x-z_j)) \neq 0$, then necessarily $|x-z_j| < 1/m_n \leq c_o/2n$ in view of (12). Thus the functions $G(m_n(x-z_j))$, $j = 1,\dots,q$, have disjoint supports so that (4) follows with $\|g_n\|_\infty = 1$. Indeed $g_n(z_i) = 1$ which implies $Q_n g_n = 1$ by (1). Furthermore,

$$0 \leq Qg_n \leq \sum_{j=1}^{q} \int_{z_j-1/m_n}^{z_j+1/m_n} G(m_n(x-z_j))dx = (q/m_n)\int_{-1}^{1} G(y)dy \leq \int_{-1}^{1} G(y)dy,$$

the latter by (12). This delivers $R_n g_n \geq 1 - \int_{-1}^{1} G(y)dy > 0$, thus (6) since

$$\omega_1(g_n,1/m_n) \geq |g_n(z_1) - g_n(z_1+1/m_n)| = 1 .$$

Condition (5) holds true in view of (cf. [5, p. 23])

$$\tau_1(g_n,\delta) \leq \delta\|g_n'\|_1 \leq \delta\, ns \int_{-1}^{1} |G'(y)|dy = M\delta n \quad ,$$

hence $\sigma(\delta) = \delta$, $\varphi_n^i = 1/n$, $i = 1,2$. Applying Theorem 4 the result follows by (9, 10, 13).

<u>Proof of Theorem 3:</u> Let X, G, and I be as before. Suppose now $\{m_{n_1}\}$ to be a subsequence satisfying (12) such that nevertheless $m_{n_1} = O(n)$. Set $T_n^1 f = R_n f$, $T_n^2 f = \omega_k(f,k/n)$, $V_n f = n\tau_k(f,1/n)$, $U_\delta f = \tau_k(f,\delta)$, $h_n^1 = g_n$ (cf. (14)), and $h_n^2(x) = G(n(x-z_1))$. As before (4) and $T_n^1 h_n^1 \geq C > 0$ hold true as well as

$$T_n^2 h_n^2 = \omega_k(h_n^2,k/n) \geq |\Delta_{1/n}^k h_n^2(z_1)| = 1$$

which delivers (6). Moreover (cf. [5, p. 24]),

$$\tau_k(h_n^1,\delta) \leq C\,\delta^k\|(h_n^1)^{(k)}\|_1 \leq C\delta^k m_n^{k-1}\int_{-1}^{1} |G^{(k)}(y)|dy = C\,\delta^k n^k ,$$

$$\tau_k(h_n^2,\delta) \leq C\,\delta^k n^{k-1} \int_{-1}^{1} |G^{(k)}(y)|dy \leq C\,\delta^k n^{k-1},$$

thus (5) with $\sigma(\delta) = \delta^k$, $\varphi_n^1 = n^{-k}$, $\varphi_n^2 = n^{-k+1}$. This also verifies (7). Finally, in view of (2)

$$V_n h_j^2 = n\,\tau_k(h_j^2,1/n) \leq nCn^{-k}\,\|(h_j^i)^{(k)}\|_1 = \mathcal{O}(\varphi_n^2) = o(\omega(\varphi_n^2))$$

which delivers (8) so that Theorem 4 proves the assertion.

References

1. H. Brass. Quadraturverfahren. Vandenhoeck & Ruprecht, Göttingen - Zürich,1977.

2. W. Dickmeis, R.J. Nessel, E. van Wickeren. Quantitative extensions of the uniform boundedness principle. Jahresber. Deutsch. Math.-Verein., 89, 1987, 105-134.

3. K.G. Ivanov. New estimates of errors of quadrature formulae, formulae of numerical differentiation and interpolation. Anal. Math., 6, 1980, 281-303.

4. R.J. Nessel, E. van Wickeren. Negative results in connection with Favard's problem on the comparison of approximation processes. In: Anniversary Volume on Approximation Theory and Functional Analysis (Proc. Conf. Oberwolfach 1983, Eds. P.L. Butzer et al.). Birkhäuser, Basel, 1984, 189-200.

5. Bl. Sendov, V.A. Popov. Averaged Moduli of Smoothness (Bulgarian). Publ. House Bulg. Acad. Sci., Sofia, 1983.

6. E. van Wickeren. A Baire approach to quantitative resonance principles. Numer. Funct. Anal. Optim., 9, 1987, 147-180.

Lehrstuhl A für Mathematik
RWTH Aachen
Templergraben 55
5100 Aachen
FRG