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AN EXTREMAL PROBLEM FOR POLYNOMIALS WITH NONNEGATIVE COEFFICIENTS. III

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1. Introduction. Let W_n be the set of all algebraic polynomials of exact degree n, all coefficients of which are nonnegative, i.e.,

$$W_n = \{P_n \mid P_n(x) = \sum_{k=0}^{n} a_k x^k,\ a_k \geq 0\,(k=0,1,\ldots,n-1),\ a_n > 0\}.$$

Let $\|f\|^2 = (f,f)$, where

$$(f,g) = \int_0^\infty w(x)f(x)g(x)\,dx \qquad (f,g \in L^2[0,\infty)),$$

with generalized Laguerre weight function $w(x)=x^\alpha e^{-x}$ $(\alpha > -1)$.

In a previous paper [2] G.V. Milovanović found a complete solution of the following problem of A.K. Varma [5]: Determine the best constant in the inequality

$$\|P_n'\|^2 \leq C_n(\alpha)\|P_n\|^2 \qquad (P_n \in W_n),$$

i.e.,

$$C_n(\alpha) = \sup_{P_n \in W_n} \frac{\|P_n'\|^2}{\|P_n\|^2}.$$

Also, G.V. Milovanović and R.Ž. Đorđević [4] considered a similar problem for Freud's weight function $w(x) = x^\alpha \exp(-x^s)$ $(\alpha > -1,\ s > 0)$ on $[0,\infty)$. A survey about extremal problems of Markov's type for algebraic polynomials is given in [3].

In this paper we consider an extremal problem for higher derivatives of polynomials

$$(1.1)\qquad C_{n,m}(\alpha) = \sup_{P_n\in W_n} \frac{\|P_n^{(m)}\|^2}{\|P_n\|^2} \qquad (1\le m\le n),$$

with respect to generalized Laguerre weight function.

The subset of W_n for which $a_0 = \dots = a_{m-1} = 0$ (i.e. $P_n(0) = \dots = P_n^{(m-1)} = 0$) we denote by W_n^{m-1}. Note that the supremum in (1.1) is attained for some $P_n \in W_n^{m-1}$. Indeed,

$$\sup_{P_n\in W_n} \frac{\|P_n^{(m)}\|}{\|P_n\|} = \sup_{\substack{P_n\in W_n^{m-1}\\ a_0,\dots,a_{m-1}\ge 0}} \frac{\|P_n^{(m)}\|}{\|P_n+Q_{m-1}\|} = \sup_{P_n\in W_n^{m-1}} \frac{\|P_n^{(m)}\|}{\|P_n\|},$$

where $Q_{m-1}(x) = \sum_{k=0}^{m-1} a_k x^k \ (a_k \ge 0)$.

2. Main result. At first, we define the integrals

$$J_k(\alpha) = \int_0^\infty x^\alpha e^{-x} P_n^{(k)}(x)^2 dx, \qquad k=0,1,\dots,m,$$

where $P_n \in W_n^{m-1}$. Using Lemma 3 from [2] we can conclude that for $\alpha > 2k-2m-1$ the following inequalities hold

$$4J_k(\alpha) \le J_{k-1}(\alpha) + (1-2\alpha)J_{k-1}(\alpha-1) + (\alpha-1)^2 J_{k-1}(\alpha-2), \quad k=1,\dots,m.$$

From these inequalities the following result follows:

Lemma 1. *The coefficients* $\beta_i^{(k)}$ *in the inequalities*

$$4^k J_k(\alpha) \le \sum_{i=0}^{2k} \beta_i^{(k)} J_0(\alpha-i), \qquad k=0,1,\dots,m,$$

satisfy the following recurrence relations

$$\beta_0^{(k)} = \beta_0^{(k-1)}, \quad \beta_1^{(k)} = \beta_1^{(k-1)} + (1-2\alpha)\beta_0^{(k-1)},$$

$$\beta_i^{(k)} = \beta_i^{(k-1)} + (1-2\alpha)\beta_{i-1}^{(k-1)} + (1-\alpha)^2\beta_{i-2}^{(k-1)}, \quad i=2,\dots,2k-2,$$

$$\beta_{2k-1}^{(k)} = (1-2\alpha)\beta_{2k-2}^{(k-1)} + (1-\alpha)^2\beta_{2k-3}^{(k-1)}, \quad \beta_{2k}^{(k)} = (1-\alpha)^2\beta_{2k-2}^{(k-1)}.$$

Lemma 2. *If the coefficients* $\beta_i^{(m)}$ *are as in Lemma 1, the following identity*

$$\sum_{i=0}^{2m} \frac{\beta_i^{(m)}}{(k+\alpha)^{(i)}} = \frac{k^2(k-2)^2\dots(k-2m+2)^2}{(k+\alpha)^{(2m)}} \qquad (\alpha > -1,\ k \geq 2m)$$

holds, where $p^{(s)} = p(p-1)\dots(p-s+1)$.

Proof of this lemma can be given by the mathematical induction.

Remark. If we define $\alpha \mapsto g(\alpha) = (\alpha-1)^2(\alpha-3)^2\dots(\alpha-2m+1)^2$, the coefficients $\beta_i^{(m)}$ can be expresed in the form

$$\beta_{2m-i}^{(m)} = \frac{(-1)^i}{i!}\Delta^i g(\alpha), \qquad i=0,1,\dots,2m,$$

where Δ is the standard forward difference operator.

Theorem. *The best constant* $C_{n,m}(\alpha)$ *defined in (1.1) is*

$$C_{n,m}(\alpha) = \begin{cases} \dfrac{(m!)^2}{(2m+\alpha)^{(2m)}}, & -1 < \alpha \leq \alpha_{n,m}, \\[2ex] \dfrac{n^2(n-1)^2\dots(n-m+1)^2}{(2n+\alpha)^{(2m)}}, & \alpha \geq \alpha_{n,m}, \end{cases} \tag{2.1}$$

where $\alpha_{n,m}$ *is the unique positive root of the equation*

$$\frac{(2n+\alpha)^{(2m)}}{(2m+\alpha)^{(2m)}} = \binom{n}{m}^2. \tag{2.2}$$

Proof. Let $P_n \in W_n^{m-1}$, i.e., $P_n(x) = \sum_{k=m}^{n} a_k x^k$ ($a_n > 0$ and other $a_k \geq 0$). Then

$$P_n(x)^2 = \sum_{k=2m}^{2n} b_k x^k \qquad (b_{2n} > 0 \text{ and other } b_k \geq 0)$$

and

$$J_0(\alpha) = \|P_n\|^2 = \sum_{k=2m}^{2n} b_k \Gamma(k+\alpha+1),$$

where Γ is the gamma function. Using Lemma 1, for k=m, we obtain

$$4^m J_m(\alpha) \le \sum_{k=2m}^{2n} b_k \left(\sum_{i=0}^{2m} \beta_i^{(m)} \Gamma(k+\alpha-i+1) \right),$$

i.e.,

$$J_m(\alpha) \le \sum_{k=2m}^{2n} H_{k,m}(\alpha) b_k \Gamma(k+\alpha+1), \tag{2.3}$$

where

$$H_{k,m}(\alpha) = \frac{1}{4^m} \sum_{i=0}^{2m} \beta_i^{(m)} \frac{\Gamma(k+\alpha-i+1)}{\Gamma(k+\alpha+1)} = \frac{1}{4^m} \sum_{i=0}^{2m} \frac{\beta_i^{(m)}}{(k+\alpha)^{(i)}},$$

or, because of Lemma 2,

$$H_{k,m}(\alpha) = \frac{k^2(k-2)^2 \ldots (k-2m+2)^2}{4^m (k+\alpha)^{(2m)}}.$$

From (2.3) it follows that

$$\|P_n^{(m)}\|^2 \le \left(\max_{2m \le k \le 2n} H_{k,m}(\alpha) \right) \|P_n\|^2$$

and so we have

$$C_{n,m}(\alpha) \le \max_{2m \le k \le 2n} H_{k,m}(\alpha),$$

where

$$\max_{2m \le k \le 2n} H_{k,m}(\alpha) = \begin{cases} H_{2m,m}(\alpha) & \text{if } -1 < \alpha \le \alpha_{n,m}, \\ H_{2n,m}(\alpha) & \text{if } \alpha \ge \alpha_{n,m}, \end{cases}$$

and $\alpha_{n,m}$ is the unique positive root of the equation (2.2).

In order to show that $C_{n,m}(\alpha)$ defined in (2.1) is best possible, i.e. that $C_{n,m}(\alpha) = \max_{2m \le k \le 2n} H_{k,m}(\alpha)$, we consider $\tilde{P}_n(x) = x^n + \lambda x^m$ ($\lambda \ge 0$) and set $Q_{n,m}(\lambda) = \|\tilde{P}^{(m)}\|^2 / \|\tilde{P}_n\|^2$. Since $Q_{n,m}(0) = H_{2n,m}(\alpha)$ and

$\lim_{\lambda\to\infty} Q_{n,m}(\lambda) = H_{2m,m}(\alpha)$, we conclude that $\tilde{P}_n(x) = x^n$ is an extremal polynomial for $\alpha \ge \alpha_{n,m}$. If $-1 < \alpha \le \alpha_{n,m}$, there exists a sequence of polynomials, for example, $p_{n,k}(x) = x^n + kx^m$, $k=1,2,\ldots$, for which

$$\lim_{k\to\infty} \|p_{n,k}^{(m)}\|^2/\|p_{n,k}\|^2 = C_{n,m}(\alpha).$$

The case m=1, where

$$(2.4) \qquad C_{n,1}(\alpha) = \begin{cases} \dfrac{1}{(2+\alpha)(1+\alpha)}, & -1 < \alpha \le \alpha_n, \\[2ex] \dfrac{n^2}{(2n+\alpha)(2n+\alpha-1)}, & \alpha_n \le \alpha < \infty, \end{cases}$$

and

$$(2.5) \qquad \alpha_n = \alpha_{n,1} = \frac{1}{2(n+1)}\left((17n^2+2n+1)^{1/2} - 3n + 1\right),$$

was considered in [2].

Note that $C_{n,m}(\alpha)$ can be found by (2.4) and (2.5) as

$$C_{n,m}(\alpha) = C_{n,1}(\alpha)C_{n-1,1}(\alpha)\ldots C_{n-m+1,1}(\alpha),$$

but only for

$$(2.6) \qquad \alpha \ge \max\{\alpha_n, \alpha_{n-1}, \ldots, \alpha_{n-m+1}\} = \alpha_{n-m+1}.$$

(The sequence (α_n) is decreasing).

<u>3. Some considerations about roots $\alpha_{n,m}$</u>. In this section we consider the equation (2.2). For m=1, the root of this equation is given by (2.5). If we put m:=n-m in (2.2), we see that $\alpha_{n,n-m} = \alpha_{n,m}$. For example, $\alpha_{n,m-1} = \alpha_{n,1} = \alpha_n$. So we will only investigate the cases when $2 \le m \le [(n+1)/2]$. Let

$$f(\alpha) = \frac{(2n+\alpha)^{(2m)}}{(2m+\alpha)^{(2m)}} = \prod_{k=1}^{2m} \frac{\alpha+k+2n-2m}{\alpha+k} \qquad (\alpha > -1).$$

Since

$$f'(\alpha) = -2(n-m)f(\alpha)\sum_{k=1}^{2m} \frac{1}{(\alpha+k)(\alpha+k+2n-2m)} < 0$$

for all $\alpha > -1$, the function $f(\alpha)$ is decreasing.

Firstly, let n=3 and m=2. Then we have $f(1/2) = \frac{143}{15} > \binom{3}{2}^2 = 9$, that means $\alpha_{3,2} > 1/2$.

Now, we consider a case when $n \geq 4$ and $m \geq 2$. Since

$$f(1/2) = \frac{(4n+1)!!}{(4m+1)!!\,(4(n-m)+1)!!} = \frac{4n+1}{(4m+1)(4(n-m)+1)} \cdot \frac{\binom{4n}{2n}\binom{2n}{n}\binom{n}{m}^2}{\binom{4m}{2m}\binom{2m}{m}\binom{4(n-m)}{2(n-m)}\binom{2(n-m)}{n-m}},$$

using improved Wallis' inequality [1]

$$\frac{4^n}{\sqrt{\pi(n+\frac{1}{2})}} < \binom{2n}{n} < \frac{4^n}{\sqrt{\pi(n+\frac{1}{4})}},$$

we obtain

$$f(1/2) > \frac{\pi}{8}\binom{n}{m}^2 \sqrt{h(4m)h(4n-4m)h(2n)},$$

where $h(x) = (2x+1)/(x+1)$. For $2 \leq m \leq [(n+1)/2]$ and $n \geq 4$ we have

$$h(4m)h(4n-4m)h(2n) \geq h(8)h(4n-8)h(2n) \geq h(8)^3 = (17/9)^3.$$

Since $\frac{\pi}{8}(17/9)^{3/2} = 1.019\ldots > 1$, we get that $f(1/2) > \binom{n}{m}^2$, that means $\alpha_{n,m} > 1/2$. On the other hand, because of (2.6), we can conclude that $\alpha_{n,m} \leq \alpha_{n-m+1}$. So we have

$$\frac{1}{2} < \alpha_{n,m} \leq \alpha_{n-m+1}.$$

In the special case, when $n \to +\infty$, we have

$$\lim_{n\to\infty} C_{n,m}(\alpha) = \begin{cases} \dfrac{(m!)^2}{(\alpha+1)_{2m}}, & -1 < \alpha \leq \alpha_m^*, \\[2ex] \dfrac{1}{4^m}, & \alpha \geq \alpha_m^*, \end{cases}$$

where $(p)_s = p(p+1)\ldots(p+s-1)$ and α_m^* is the unique root of equation $(\alpha+1)_{2m} = 4^m(m!)^2$.

We note that $\alpha_1^* = \alpha_\infty = (\sqrt{17}-3)/2$.

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