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NUMERICAL TREATMENT OF TWO-POINT BOUNDARY VALUE PROBLEMS WITH SPLINE FUNCTIONS

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1. Introduction. This paper is concerned with the numerical solution of the boundary value problem

$$y''(x) = f(x,y(x)), \quad x \in [a,b]$$
$$(1.1) \qquad y(a) = A$$
$$y(b) = B$$

where f is a continuous real-valued bivariate function. Moreover assume that there exists a constant L such that for every $x \in [a,b]$, $y,y' \in \mathbb{R}$

$$|f(x,y) - f(x,y')| \leq L \cdot |y-y'|.$$

We suppose that (1.1) has unique solution $u \in C^2[a,b]$.

We first outline some of the notations used in this paper: Let the finite interval [a,b] be partitioned into N+1 equal parts by the insertion of N knots x_i defined by $x_i = a+i\cdot h$, $h = (b-a)/(N+1)$, $i = 0, \ldots, N+1$. Let π_m denote the space of polynomials of degree at most m ($m \geq 3$) and let $S_m(\Delta)$ be the space of polynomial spline functions of degree m with simple knots x_i ($i = 1, \ldots, N$). The B-splines associated to the given knot partition Δ are defined by

$$B_{m,i}(x) := \frac{1}{h^m \cdot m!} \sum_{\nu=0}^{m+1} (-1)^{\nu} \binom{m+1}{\nu} (x-x_{i+\nu})_+^m .$$

It is well-known that the functions $B_{m,i}$ $(i=-m,\ldots,N)$ form a basis of $S_m(\Delta)$ with relatively small support and can by computed by a stable recurrence relation.

Looking for a solution of (1.1) we use a collocation method by spline functions, i.e. we determine a function $s \in S_m(\Delta)$ with the properties

$$(1.2)\qquad \begin{aligned} s''(x_i) &= f(x_i, s(x_i)) \qquad i=0,\ldots,N+1 \\ s(a) &= A \\ s(b) &= B. \end{aligned}$$

There is a lot of papers concerning with such collocation methods. Important results has been given for example by Ascher, De Boor, Swartz Christiansen and Pruess (see [2],[4],[5],[9]). These authors investigated more general boundary value problems (of order k) using arguments of functional analysis and working with splines having knots of multiplicity m-k. Another approach for solving (1.2), which leads to quantitative more precise results, was carried out by Ahlberg and Ito [1], Usmani [10], Khalifa and Eilbeck [7]. Using methods of the linear algebra they transformed (1.2) into a nonlinear system of equations and obtained existence, uniqueness and convergence results, but only for splines of degree at most seven. In [8] we were able to extend this method to spline functions of any degree. The purpose of this article is to give some of the results obtained in that paper.

2. Development of the main recurrence relations. In this section we transform the collocation and boundary conditions into a finite difference scheme which contains the values $s(x_i)$ $(i=1,\ldots,N)$ as unknowns.

For simplicity the following notation is introduced:

$$b_{m,l}^{(\nu)} := B_{m,i}^{(\nu)}(x_{i+l}), \quad b_{m,l} := b_{m,l}^{(0)} = B_{m,i}(x_{i+l}).$$

We now state the main result of this section.

Lemma 2.1. Let m be a positive integer, $m \geq 3$, and $s \in S_m(\Delta)$. Then the following recurrence relations hold:

$$(2.1)\qquad \sum_{l=1}^{m} b_{m,l}\, s''(x_{i+l}) = \sum_{l=1}^{m} b''_{m,l}\, s(x_{i+l}) \qquad (i=-1,\ldots,N-m+1).$$

If m is even, $m \geq 4$, then

$$(2.2) \qquad \sum_{\mu=1}^{m-1} \sum_{\sigma=1}^{\mu} (-1)^{\mu-\sigma} b_{m,\sigma} s''(x_{i+\mu}) = \sum_{\mu=1}^{m-1} \sum_{\sigma=1}^{\mu} (-1)^{\mu-\sigma} b''_{m-\sigma} s(x_{i+\mu}) \qquad (i=-1,\ldots,N-m+2).$$

The proof of this statement can be found in [8, p.10].

Example. For $m=5$ we obtain from (2.1) a formula which was developed by Usmani [10],

$$s(x_{i-2}) + 2\,s(x_{i-1}) - 6\,s(x_i) + 2\,s(x_{i+1}) + s(x_{i+2}) =$$

$$= h^2\,(s''(x_{i-2}) + 26\,s''(x_{i-1}) + 66\,s''(x_i) + 26\,s''(x_{i+1}) + s''(x_{i+2}))/20 \qquad (i=2,\ldots,N-1).$$

Replacing $s''(x_{i+l})$ by $f(x_{i+l}, s(x_{i+l}))$ the relations (2.1) and (2.2) can be used for collocation with splines. For example, if m is odd we get

$$(2.3) \qquad \sum_{l=1}^{m} b''_{m,l}\, s(x_{l+\nu-1}) = \sum_{l=1}^{m} b_{m,l}\, f(x_{l+\nu-1}, s(x_{l+\nu-1})) \qquad (\nu=0,\ldots,N-m+2).$$

The corresponding difference operator (D.O) is defined by

$$L_0^{(m)}(y(x);h) := \sum_{l=1}^{m} h^2 b''_{m,l}\, y(x+(l-1)h) - h^2 \cdot \sum_{l=1}^{m} b_{m,l}\, y''(x+(l-1)h).$$

Lemma 2.2. The order p of $L_0^{(m)}$ is at least m-1 if is odd and at least m if m is even.

This lemma was verified in [8, p. 42].

For $m=3$ the number of boundary and collocation conditions in (1.2) coincides with the dimension of $S_m(\Delta)$. But for $m>3$ proper additional conditions are needed in order to obtain uniqueness of the solution $s \in S_m(\Delta)$ in (1.2). These additional conditions will be established such that

- the matrix of the coefficients which is derived from the collocation and additional conditions has banded structure.

- the order of the difference operator associated to the additional conditions is at least p.

For this purpose we choose an appropriate linear combination of Cowell's formulas

$$y_{\nu-2}-2y_{\nu-1}+y_\nu = h^2 \sum_{\mu=0}^{\tilde{m}-1} \sigma_\mu^* \Delta^\mu M_{\nu-2} \quad (\nu=2,3,\ldots),$$

$$y_{\nu-2}-2y_{\nu-1}+y_\nu = h^2 \cdot \sum_{\mu=0}^{\tilde{m}-1} \sigma_\mu^* \nabla_\mu M_\nu \quad (\nu=N+1,N,\ldots)$$

where

$$\tilde{m} := \begin{cases} m-1 & \text{if } m \text{ is odd} \\ m & \text{if } m \text{ is even,} \end{cases} \qquad M_\nu := f(x_\nu,y_\nu),\ y_\nu := s(x_\nu),$$

$$\Delta^\mu M_\nu := \sum_{\sigma=0}^{\mu} (-1)^\sigma \binom{\mu}{\sigma} M_{\nu+\sigma},\ \Delta^\mu M_\nu := \sum_{\sigma=0}^{\mu} (-1)^\sigma \binom{\mu}{\sigma} M_{\nu-\sigma} \quad (\mu \in \mathbb{N}_0),$$

$$\sigma_\mu^* := (-1)^\mu \int_{-1}^{0} (-s) \left[\binom{-s}{\mu} + \binom{s+2}{\mu} \right] ds \ (\mu=0,\ldots,\tilde{m}-1).$$

It is well-known (Henrici [6, p. 297]) that the order of the associated D.O. of the Cowell's formula is exakt $\tilde{m}$.

After these preparations we can explicitely give the additional conditions. For the sake of simplicity we do this only for the case when m is odd:

(2.4a)
$$\sum_{\nu=0}^{s+\mu-2} (y_\nu-2y_{\nu-1}+y_{\nu+2}) b_{m-2,s+\mu-\nu-1} -$$
$$\sum_{\nu=0}^{s-\mu-2} (y_\nu-2y_{\nu+1}+y_{\nu+2}) b_{m-2,s-\mu-\nu-1} =$$
$$h^2 \sum_{\nu=0}^{s+\mu-2} \left\{ \sum_{j=0}^{m-2} \sigma_j^* \Delta^j M_\nu \right\} b_{m-2,s+\mu-\nu-1} -$$
$$h^2 \sum_{\nu=0}^{s-\mu-2} \left\{ \sum_{j=0}^{m-2} \sigma_j^* \Delta^j M_\nu \right\} b_{m-2,s-\mu-\nu-1} \quad (\mu=1,\ldots,s-1),$$

(2.4b)
$$\sum_{\nu=\mu-s+2}^{N+1} (y_\nu-2y_{\nu-1}+y_{\nu-2}) b_{m-2,s-\mu+\nu-1} -$$
$$\sum_{\nu=2N-s-\mu+4}^{N+1} (y_\nu-2y_{\nu-1}+y_{\nu-2})\ b_{m-2,-2N+s+\mu+\nu-3} =$$

$$h^2 \sum_{\nu=\mu-s+2}^{N+1} \left\{ \sum_{j=0}^{m-2} \sigma_j^* \nabla^j M_\nu \right\} b_{m-2,s-\mu+\nu-1} -$$

$$- h^2 \sum_{\nu=2N-s-\mu+4}^{N+1} \left\{ \sum_{j=0}^{m-2} \sigma_j^* \nabla^j M_\nu \right\} b_{m-2,-2N+s+\mu+\nu-3} \quad (\mu=N-s+2,\dots,N)$$

where $s = (m-1)/2$.

Example. For $m = 5$ we obtain the formulas (Usmani [10, p. 158])

$$4y_0-7y_1+2y_2+y_3 = h^2(4M_0+41M_1+14M_2+M_3)/12,$$

$$y_{N-2}+2y_{N-1}-7y_N+4y_{N+1} = h^2(M_{N-2}+14M_{N-1}+41M_N+4M_{N+1})/12.$$

We summarize boundary, collocation and additional conditions (see (1.2),(2.3),(2.4a),(2.4b)) in a nonlinear system of equations

(2.5) $$\varphi(y) := -D^{(m)}y + h^2 R^{(m)} f(y) + v^{(m)} = 0,$$

$$y = (y_1,\dots,y_N) = (s(x_1),\dots,s(x_N)), f(y) = (M_1,\dots,M_N).$$

3. Existence and Convergence results.

We need the following theorem of Newton Kantorovich:

Theorem 3.1. Assume that φ is a map of the form (2.5) and $y^{(0)}$ is a given initial vector. Furthermore let the following conditions be satisfied:

(1) The matrix $(\varphi'(y^{(0)}))^{-1}$ exists and an estimate for its norm is known.

(3.1) $$\|\varphi'(y^{(0)})^{-1}\|_\infty \le \sigma_1.$$

(2) There exists a constant σ_2, such that for all y in the region defined by (3.5) below the following estimation is satisfied:

(3.2) $$\sum_{j,k=1}^{N} \left| \frac{\partial^2 \varphi_i}{\partial y_j \, \partial y_k}(y) \right| \le \sigma_2 \quad (i=1,\dots,N).$$

(3) The vektor $y^{(0)}$ approximately satisfies the system of equations (2.5) in the sense that

(3.3) $$\|\varphi'(y^{(0)})^{-1}\varphi(y^{(0)})\|_\infty \le \sigma_3.$$

(4) The introduced constants σ_1, σ_2 and σ_3 satisfy the inequality

$$\eta := \sigma_1 \sigma_2 \sigma_3 \leq \frac{1}{2}. \tag{3.4}$$

Then the system of equations (2.5) has a unique solution y* which is located in the cube

$$\|y - y^{(0)}\|_\infty \leq (1 - \sqrt{1-2\eta}) \cdot \sigma_3 / \eta. \tag{3.5}$$

Moreover, Newton's method defined by

$$y^{(n+1)} = y^{(n)} - \varphi'(y^{(n)})^{-1} \varphi(y^{(n)}) \qquad (n \in \mathbb{N}_0) \tag{3.6}$$

converges quadratically to y*.

For the proof see Henrici [6, p. 367].

We now succesively fulfill the conditions (1) - (4) of Theorem 3.1. Obviously we have $\varphi'(y) = -D^{(m)} + h^2 R^{(m)} F(y)$, where $F(y) = \text{diag}\left(\frac{\partial f}{\partial y}(x_1, y_1), \ldots, \frac{\partial f}{\partial y}(x_N, y_N)\right)$. Because of the special choice of the additional conditions it is possible to factorize the banded matrix $D^{(m)}$ in a product of tridiagonal matrices.

Lemma 3.2. Let $s = (m-1)/2$ if m is odd and $s = (m-2)/2$ if m is even. Then

(1) there exist constants $\beta_1, \ldots, \beta_s \in \mathbb{R}$, $\beta_1 = -2$, such that

$$D^{(m)} = \prod_{k=1}^{s} Q^{(k)} \tag{3.7}$$

where $Q_{ii}^{(k)} = \beta_i$, $Q_{i,i+1}^{(k)} = Q_{i+1,i}^{(k)} = 1$ $(i=1, \ldots, N-1)$ and $Q_{ij}^{(k)} = 0$ elsewhere.

(2) the matrices $Q^{(k)}$ defined above are regular and the following estimations hold.

$$\|(Q^{(1)})^{-1}\|_\infty \leq (N+1)^2/8, \quad \|(Q^{(i)})^{-1}\|_\infty \leq 1/(\beta_i - 2) \quad (i=2,\ldots,s)$$

(3) $$(D^{(m)})^{-1} = \sum_{i=1}^{s} \alpha_i (Q^{(i)})^{-1}, \quad \alpha_i = \prod_{\substack{j=1 \\ j \neq i}}^{s} (\beta_j - \beta_i)^{-1}$$

The proof of this statement was given in [8, p. 29-39].

It follows from Lemma 3.2 that

$$(3.8) \qquad \|(D^{(m)})^{-1}\|_\infty \le \frac{(b-a)^2}{\xi h^2 (m-2)!} + \sum_{i=2}^{s} \frac{\alpha_i}{\beta_i - 2}$$

where $\xi = 8$ if m is odd and $\xi = 4$ if m is even.

Applying a theorem of ATKINSON ([3,p. 425]) we can easily show

Lemma 3.3. Assume that

$$(3.9) \qquad c_1 := \alpha_1 (b-a)^2 L \|R^{(m)}\|_\infty / 8 < 1,$$

$$(3.10) \qquad h^2 < (1-c_1)/(\|R^{(m)}\|_\infty \, L \, c_2),$$

$$c_2 := \sum_{i=2}^{s} \alpha_i/(\beta_i - 2).$$

Then the following estimation holds:

$$(3.11) \qquad \|\varphi'(y)^{-1}\|_\infty \le \frac{\alpha_1 (b-a)^2/(8h^2) + c_2}{1 - c_1 - h^2 L \|R^{(m)}\|_\infty \, c_2} =: \sigma_1.$$

Remarks. (1) If m is odd, then $\alpha_1 = 1/(m-2)!$; if $m \in \{2,4,6\}$, then $\alpha_1 = 2/(m-2)!$.

(2) For $m \le 7$ we have $\|R^{(m)}\|_\infty = 1$.

Straightforward analysis leads to the two other estimations which occur in the theorem of Newton-Kantorovich:

$$(3.12) \qquad \sum_{j,k=1}^{N} \left| \frac{\partial \varphi_\mu}{\partial y_j \, \partial y_k} (y) \right| \le h^2 \cdot L_2 \cdot \|R^{(m)}\|_\infty =: \sigma_2$$

where $L_2 := \max \{ |f_{yy}(x,y)| : x \in [a,b], \ y \in \mathbb{R} \}$ is assumed to be finite. Let the initial vector defined by $y_\nu^{(0)} := z(x_\nu)$, $\nu = 1, \ldots, N$ where $z(x)$ is a $(\tilde{m}+2)$-times differentiable function satisfying $z(a) = A$, $z(b) = B$. Define $Z^{(m)} := \|z^{(\tilde{m}+2)}\|_\infty$, $R := \max \{ |z''(x) - f(x, z(x))| : x \in [a,b] \}$

and $S^{(m)} := \max \{ (m-2)!, \max_\mu \sum_{\nu=0}^{N+1} |\beta_{\mu\nu}^{(m)}| \}$, where $\beta_{\mu\nu}^{(m)}$ are the coefficients which occur in the additional conditions.

Then there exists a constant $G^{(m)}$ which can be explicitely computed, see [8,p.44-45], such that

$$(3.13) \qquad \|\varphi'(y^{(0)})^{-1}\varphi(y^{(0)})\|_\infty \leq \sigma_1 \cdot [h^2 RS^{(m)} + h^{m+2} G^{(m)} Z^{(m)}] =: \sigma_3.$$

We summarize the results in the following

Theorem 3.4. Let $u \in C^{\tilde{m}+2}[a,b]$ be the exact solution of the differential equation (1.1) and let the conditions (3.9) and (3.10) be satisfied. If $\sigma_1 \cdot \sigma_2 \cdot \sigma_3 \leq 1/2$, then the system of equation (2.5) has a unique solution y^* and the sequence $y^{(n)}$, defined by (3.6) converges quadratically to y^*.

We remark that the uniqueness of y^* follows from the Lipschitz-continuity of f and from the regularity of $D^{(m)}$, existence and convergence from Theorem 3.1.

Example. If $m \leq 5$, $z \in \pi_{\tilde{m}+1}$ and h is sufficiently small, then existence, uniqueness and convergence results are guaranteed, if

$$(b-a)^4 \cdot L_2 \cdot R < \begin{cases} 32 & \text{for } m=3 \text{ or } m=4 \\ 192 & \text{for } m=5. \end{cases}$$

Under quite weak assumptions (for details see [8]), it is possible to show that the spline s determined by (1.2) and by the additional conditions converges to the exact solution u of (1.1). Furthermore the following estimations hold:

$$\max_{1 \leq i \leq n} |s(x_i) - u(x_i)| = \begin{cases} O(h^{m-1}) & \text{if } m \text{ is odd} \\ O(h^m) & \text{if } m \text{ is even.} \end{cases}$$

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