CONSTRUCTIVE THEORY OF FUNCTIONS'87

SOFIA, 1988

SATURATION OF A NEW METHOD OF KANTOROVIČ TYPE MEYER - KÖNIG AND ZELLER OPERATORS

Manfred W. Müller

1. Introduction. Let $(\phi_n)_{n\in N}$ be a sequence of functions $\phi_n: [o,r] \to \mathbb{R}$ $(r > o)$ with the properties (i) $\phi_n \in C^\infty[o,r]$, (ii) $\phi_n(o) = 1$, (iii) $(-1)^k \phi_n^{(k)}(x) \geq o$, $k \in \mathbb{N}_o$, $x \in [o,r]$, (iv) there exists an integer c such that $\phi_n^{(k)}(x) = -n\, \phi_{n-c}^{(k-1)}(x)$, $k \in \mathbb{N}_o$, $n > \max(c,o)$, $x \in [o,r]$. Based on these functions Lehnhoff [4] introduced in 1979 a sequence of positive linear operators $M_{n,q}$ (depending on a parameter $q \in \mathbb{N}_o$), which he calls operators of Meyer-König and Zeller type, by

$$(1)\qquad (M_{n,q}f)(x) := \sum_{k=o}^{\infty} \frac{(-1)^k}{k!} \phi_{n+q}^{(k)}\left(\frac{x}{1-x}\right)\left(\frac{x}{1-x}\right)^k f\left(\frac{k}{k+n}\right), \quad x \in [o, \tfrac{r}{r+1}],$$

for all $f \in C[o,1)$ for which the series converge.

A suitable sequence of functions solving (i) - (iv) for $c = -1$ is $\phi_n(x) = (1+x)^{-n}$, $x \in [o,\infty)$ (i.e. $r = \infty$). (1) reduces then for $q = o$ to the original Meyer-König and Zeller operator [7] and for $q = 1$ to the Meyer-König and Zeller operator in the modified form of Cheney and Sharma [3], both with $x \in [o,1)$ and $f \in C[o,1)$, such that the series converge.

The series $M_{n,q}f$ cannot be used for the approximation of functions $f \in L_p[o,1]$, $1 \leq p < \infty$, in the L_p-metric. According to an idee of Kantorovič we introduce therefore via

$(M_{n,q}^* f)(x) := \frac{d}{dx}(M_{n,q}F)(x)$, where F is a primitive of $f \in L_1[o,1]$,

the so-called Kantorovič type Meyer-König and Zeller operators $M_{n,q}^*$ being positive linear operators from $L_1[o,1]$ into $C[o, \frac{r}{r+1}]$ and having on account of (iv) the representation

$$(2) \qquad (M^*_{n,q} f)(x) := \frac{(n+q)}{(1-x)^2} \sum_{k=o}^{\infty} \frac{(-1)^k}{k!} \left(\frac{x}{1-x}\right)^k \phi^{(k)}_{n+q-c}\left(\frac{x}{1-x}\right) \int_{\frac{k}{k+n}}^{\frac{k+1}{k+n+1}} f(t)dt \ ,$$

$x \in [o, \frac{r}{r+1}]$. Again for the special choice of the ϕ_n as above (2) reduces for $q = 1$ to the <u>integrated Meyer-König and Zeller operator</u> with $x \in [o,1)$, introduced by the author [8] and studied in two subsequent papers [6], [9]. (Two years later this operator has been rediscovered by Totik [12] and designated as <u>Kantorovič type modification of the Meyer-König and Zeller operator</u>.) Further examples for operators of the type (2) are not known.

2. The method. The conditions (i) - (iv) are solved for $c = 1$ by the sequence of functions $\phi_n(x) = (1-x)^n$, $x \in [o,1]$ (i.e. $r = 1$). (2) reduces then for $q = o$ to

$$(3) \qquad (M^*_{n,o} f)(x) = \frac{n}{(1-x)^{n+1}} \sum_{k=o}^{n-1} \binom{n-1}{k} x^k (1-2x)^{n-1-k} \int_{\frac{k}{k+n}}^{\frac{k+1}{k+n+1}} f(t)dt \ ,$$

$x \in [o, \frac{1}{2}]$.

$M^*_{n,o}$ is a positive linear polynomial operator from $L_1[o, \frac{1}{2}]$ into $C[o, \frac{1}{2}]$ and turns out to have a close relationship to the classical Kantorovič operator

$$(P_n f)(x) = (n+1) \sum_{k=o}^{n} \binom{n}{k} x^k (1-x)^{n-k} \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t)dt \ , \ x \in [o,1]$$

for $f \in L_1[o,1]$. In fact, the substitutions $y = \frac{x}{1-x}$ and $u = \frac{t}{1-t}$ are transforming the right hand side of (3) into

$$(1+y)^2 \, n \sum_{k=o}^{n-1} \binom{n-1}{k} y^k (1-y)^{n-1-k} \int_{\frac{k}{n}}^{\frac{k+1}{n}} f\left(\frac{u}{1+u}\right) \frac{du}{(1+u)^2}$$

which means that

$$(M^*_{n,o}f)(x) = (1+y)^2(P_{n-1}g)(y) \quad ,$$

(4) $$g(u) = \frac{1}{(1+u)^2} f(\frac{u}{1+u}) \ , \quad u \in [o,1] \ .$$

Evidently $g \in L_p[o,1]$ if $f \in L_p [o, \frac{1}{2}]$ $1 \leq p < \infty$.

The key for our further investigations is the following

Lemma. For $f \in L_p[o, \frac{1}{2}]$ and the corresponding $g \in L_p[o,1]$, $1 \leq p < \infty$, there holds

(5) $$\| M^*_{n,o}f - f\|_1 [o, \frac{1}{2}] = \| P_{n-1}g - g\|_1 [o,1] \ ,$$

(6) $$\| P_{n-1}g - g\|_p [o,1] \leq \| M^*_{n,o} f - f\|_p [o,\tfrac{1}{2}] \leq 4\| P_{n-1}g - g\|_p [o,1] \quad \text{for } p>1 \ .$$

Proof. From $dx = \frac{1}{(1+y)^2} dy$ and

$$(M^*_{n,o} f)(x)-f(x) = (1+y)^2(P_{n-1}g)(y) - \frac{1}{(1+y)^2} f(\frac{y}{1+y}) \ (1+y)^2$$

$$= (1+y)^2[P_{n-1})(y)- g(y)]$$

(5) is obtained immediately. For $p > 1$ we have

$$\int_o^{\frac{1}{2}} |(M^*_{n,o} f)(x)- f(x)|^p dx = \int_o^1 |(P_{n-1}g)(y) - g(y)|^p(1 + y)^{2p-2} dy$$

$$\leq 2^{2p} \int_o^1 |(P_{n-1}g)(y) - g(y)|^p dy \ ,$$

yielding the right hand part of (6). The estimate $(1 + y)^{2p-2} \geq 1$ gives in the same way the left hand part of (6).

Corollary. $\lim\limits_{n\to\infty} \| f - M^*_{n,o} f\|_p [o, \frac{1}{2}] = o$ for every $f \in L_p[o, \frac{1}{2}]$, $1 \leq p < \infty$ (i.e. $(M^*_{n,o})_{n \in N}$ is a positive linear polynomial approximation method in the L_p - metric).

3. Global saturation. Combining the global saturation theorems for Kantorovič polynomials of Maier [5] (p = 1) and Riemenschneider/Totik [10,11] (p > 1) with the above lemma one obtains

Theorem 1. If $f \in L_p[0,\frac{1}{2}]$, $1 \le p < \infty$, then

(7) $$\| M_{n,0}^{*}f - f\|_p [0,\tfrac{1}{2}] = o(\tfrac{1}{n}) \iff f(t) = \frac{c}{(1-t)^2}, \quad c \in \mathbb{R}.$$

(8) $$\| M_{n,0}^{*}f - f\|_p [0,\tfrac{1}{2}] = O(\tfrac{1}{n})$$

$$\iff \begin{cases} f \in AC[0,\tfrac{1}{2}]^{1)}, \ h := t(1-2t)\, f' \in BV[0,\tfrac{1}{2}] \text{ with} \\ h(0) = h(\tfrac{1}{2}) = 0 \quad \text{for } p = 1; \\ f' \in AC[0,\tfrac{1}{2}], \ t(1-2t)f'' \in L_p[0,\tfrac{1}{2}] \text{ for } p > 1. \end{cases}$$

Proof. (7) follows from the equivalences

$$\| M_{n,0}^{*}f - f\|_p [0,\tfrac{1}{2}] = o(\tfrac{1}{n}) \iff \| P_{n-1}g - g\|_p [0,1] = o(\tfrac{1}{n})$$

$$\iff g(u) = c, \ c \in \mathbb{R} \iff f(t) = \frac{c}{(1-t)^2}, \ c \in \mathbb{R}.$$

(By direct calculation it can easily be shown that the functions $f(t) = \frac{c}{(1-t)^2}$ are invariants of the operators $M_{n,0}^{*}$.) In the same way

$$\| M_{n,0}^{*}f - f\|_p [0,\tfrac{1}{2}] = O(\tfrac{1}{n}) \iff \| P_{n-1}g - g\|_p [0,1] = O(\tfrac{1}{n}),$$

which on his part is equivalent to

$$\begin{cases} g \in AC[0,1], \ \eta := u(1-u)\, g' \in BV[0,1] \text{ with } \eta(0) = \eta(1) = 0 \text{ for } p = 1; \\ g' \in AC[0,1], \ u(1-u)g'' \in L_p[0,1] \text{ for } p > 1, \end{cases}$$

and this again to

$$\begin{cases} (1-t)^2 f \in AC[0,\tfrac{1}{2}], \ \bar{h} := t(1-2t)(1-t)^2 f' - 2t(1-2t)(1-t)f \in BV[0,\tfrac{1}{2}] \text{ with} \\ \bar{h}(0) = \bar{h}(\tfrac{1}{2}) = 0 \quad \text{for } p = 1; \\ (1-t)^4 f' - 2(1-t)^3 f \in AC[0,\tfrac{1}{2}], \\ t(1-2t)(1-t)^4 f'' - 6t(1-2t)(1-t)^3 f' + 6t(1-2t)(1-t)^2 f \in L_p[0,\tfrac{1}{2}] \text{ for } p > 1, \end{cases}$$

---

1) This means that f coincides a. e. with an absolutely continuous function.

which proves (8), since all the factors $(1-t)^k$ are redundant and $t(1-2t)\, f' \in BV[0,\frac{1}{2}]$ implies $f \in AC[0,\frac{1}{2}]$.

4. Local saturation. Corresponding reductions of a local saturation theorem for Kantorovič polynomials by Ditzian and May [2] lead to the following

Theorem 2. If $f \in L_p[0,\frac{1}{2}]$, $1 \le p < \infty$, and $0 < a < a_1 < b_1 < b < \frac{1}{2}$, then

(9) $$\|M_{n,0}^* f - f\|_p [a,b] = O(\frac{1}{n}) \implies f \in S_p[a,b],$$

(10) $$f \in S_p[a,b] \implies \|M_{n,0}^* f - f\|_p [a_1,b_1] = O(\frac{1}{n}),$$

(11) $$\|M_{n,0}^* f - f\|_p [a,b] = o(\frac{1}{n}) \implies f(t) = \frac{1}{(1-t)^2}[C_1 + C_2 \ln \frac{t}{1-2t}] \text{ on } [a,b],$$

(12) $$f(t) = \frac{1}{(1-t)^2}[C_1 + C_2 \ln \frac{1}{1-2t}] \text{ on } [a,b] \implies \|M_{n,0}^* f - f\|_p [a_1, b_1] = o(\frac{1}{n}),$$

where

$$S_p[a,b] := \{f \in L_p[0,\frac{1}{2}] : f' \in AC[a,b],\, f'' \in L_p[a,b]\} \text{ for } p > 1,$$

$$S_1[a,b] := \{f \in L_1[0,\frac{1}{2}] : f \in AC[a,b],\, f' \in BV[a,b]\}.$$

A characterization of the local saturation classes is given in

Theorem 3. If $f \in L_p[0,\frac{1}{2}]$, $1 \le p < \infty$, then the following statements are equivalent:

(13) $f \in S_p[a,b]$;

(14) $$\begin{cases} \|\Delta_h^* f\|_p [a+h, b-h] \le Kh^2, \quad 0 < h \le \frac{b-a}{2}, & \text{if } p > 1, \\ \|\Delta_h^* F\|_p [a+h, b-h] \le Kh^2 & \text{if } p = 1, \end{cases}$$

where Δ_h^* denotes the symmetric second difference

$\Delta_h^*(f;x) = f(x-h) - 2f(x) + f(x+h)$ and F is a primitive of f.

Proof. The direction (13) $\implies$ (14) and the case $p = 1$ can be treated as in the paper of Becker and Nessel [1]. We sketch a proof for the direction (14) $\implies$ (13) and $p > 1$. Let us consider the bilinear functional

$$L_n(f,g) := \int_a^b f_n(t) g(t)\,dt \text{ for } f \in L_p[0,\frac{1}{2}],\ g \in L_q[a,b],\ \frac{1}{p} + \frac{1}{q} = 1,$$

where

$$f_n(t) := \begin{cases} n^2 \Delta^*_{\frac{1}{n}}(f;t) & \text{for } t \in [a+\frac{1}{n}, b-\frac{1}{n}] \\ 0 & \text{otherwise} \end{cases} \qquad n \in \mathbb{N} .$$

For an arbitrary $g \in C^2[a,b]$ with compact support in $[a,b]$ one obtains easily $|L_n(f,g)| \leq C_g \|f\|_p [a,b]$ with C_g independant of $f \in L_p[o,\frac{1}{2}]$ and n. Therefore an integration by parts gives immediately for $f \in L_p[o,\frac{1}{2}]$ the weak* convergence

$$L(f,g) := \lim_{n\to\infty} L_n(f,g) = \int_a^b f(t)g''(t)dt.$$

Let now $f \in L_p[o,\frac{1}{2}]$ be fixed. On account of

$$|L_n(f,g)| \leq \|f_n\|_p [a+\frac{1}{n}, b-\frac{1}{n}] \|g\|_q [a,b] \leq K \|g\|_q [a,b],$$

for every $g \in L_q[a,b]$, the sequence of norms $\text{Sup}\{|L_n(f,g)| : \|g\|_q [a,b] = 1\}$ is uniformly bounded in n. Consequently the weak* compactness theorem implies for $g \in L_q[a,b]$ the weak* convergence

$$\bar{L}(f,g) := \lim_{k\to\infty} L_{n_k}(f,g) = \int_a^b h(t)g(t)dt$$

for a suitable $h \in L_p[a,b]$ and a subsequence (n_k). From the unicity of the limit functionals there follows now by a standard argument that

$$f'(t) = \int_a^b h(u)du \quad , \quad t \in [a,b] \quad ,$$

which completes the proof.

The nonoptimal case will be treated in a forthcoming paper.

References

1. M. Becker and R.J. Nessel. On global saturation for Kantorovich polynomials. In: Approximation and function spaces (Proc. Conf. Gdánsk, 1979). North-Holland, Amsterdam, 1981, 89-101.

2. Z. Ditzian and C.P. May. L_p-saturation and inverse theorems for modified Bernstein polynomials. Indiana Univ. Math. J. 25 (1976), 733-751.

3. E.W. Cheney and A. Sharma. Bernstein power series. Canad. J. Math. 16 (1964), 241-253.

4. H.G. Lehnhoff. Lokale Approximationsmaße und Nikolskii-Konstanten für positive lineare Operatoren. Dissertation, Universität Dortmund 1979 (140 pp.).

5. V. Maier. The L_1-saturation class of the Kantorovich operators. J. Approx. Theory 22 (1978), 223-232.

6. V. Maier, M.W. Müller, and J. Swetits. The local L_1-saturation class of the method of integrated Meyer-König and Zeller operators. J. Approx. Theory 32 (1981), 27-31.

7. W. Meyer-König und K. Zeller. Bernsteinsche Potenzreihen. Studia Math. 19 (1960), 89-94.

8. M.W. Müller. L_p-approximation by the method of integral Meyer-König and Zeller operators. Studia Math. 63 (1978), 81-88.

9. M.W. Müller und V. Maier. Die lokale L_p-Saturationsklasse des Verfahrens der integralen Meyer-König und Zeller Operatoren. In: Linear Spaces and Approximation (Proc. Conf. Oberwolfach, 1977, Eds. P.L. Butzer and B. Sz.-Nagy, ISNM 40). Birkhäuser, Basel, 1978, 305-317.

10. S.D. Riemenschneider. The L_p-saturation of the Bernstein-Kantorovich polynomials. J. Approx. Theory 23 (1978), 158-162.

11. V. Totik. $L_p(p > 1)$-approximation by Kantorovich polynomials. Analysis 3 (1983), 79-100.

12. V. Totik. Approximation by Meyer-König and Zeller type operators. Math. Zeitschrift 182 (1983), 425-446.

Universität Dortmund, Lehrstuhl Mathematik VIII, Postfach 500 500,
D-4600 Dortmund 50, BR-Deutschland