CONSTRUCTIVE THEORY OF FUNCTIONS'87

SOFIA, 1988

ON STRONG EULER ABSOLUTE φ-SUMMABILITY A.E. OF ORTHONORMAL SERIES

Helena Musielak

Generalizing the results of O.A.Ziza [3] and V.N.Spevakov [2], there is proved in [1] the following

Theorem. Let $E_n^q(x)$ be the Euler means of order $q>0$ of the sequence of partial sums of a real orthonormal series $\sum_{k=0}^{\infty} c_k \Phi_k(x)$ in $\langle 0,1\rangle$. Let $\beta<\frac{5}{2}$, $0<\gamma\leq 2$, $\gamma>\frac{4}{7}(\beta+1)$ and

$$H_n = \Big(\sum_{k=2^n+1}^{2^{n+1}} k^{\frac{2}{\gamma}(\beta+1)-\frac{3}{2}} c_k^2\Big)^{\gamma/2}.\ \text{If}\ \sum_{n=0}^{\infty} H_n<\infty,\ \text{then}$$

$$(1)\qquad \sum_{n=0}^{\infty} n^{\beta}\left|E_n^q(x)-E_{n-1}^q(x)\right|^{\gamma}<\infty \quad \text{a.e.}$$

The aim of this paper is to generalize this result, replacing the series (1) by means of $\sum_{n=0}^{\infty}\varphi_n(\lambda|E_n^q(x)-E_{n-1}^q(x)|)$, where $(\varphi_n)_{n=0}^{\infty}$ is a sequence of φ-functions satisfying suitable conditions. Writing $Z_n^q(x)=E_n^q(x)-E_{n-1}^q(x)$ and $L_{n,k}^q=\frac{1}{(1+q)^n}\binom{n}{k}q^{n-k}$, we have $Z_n^q(x)=\sum_{k=0}^{n} L_{n,k}^q\,\frac{k}{n}\,c_k\Phi_k(x)$. Moreover, $L_{n,k}^q\leq C_q n^{-1/2}$ and $\sum_{n=k}^{\infty} L_{n,k}^q\leq A_q$ for $k=0,1,2,\dots$ (see [2], p.358,359).

Let $\varphi=(\varphi_n)_{n=0}^{\infty}$ be a sequence of increasing φ-functions and let $\psi_n(u)=\left(\varphi_n^{-1}(u)\right)^2$ for $u\geq 0$, where φ_n^{-1} is the inverse function for φ_n. Thus, $\psi_n(\varphi_n(u))=u^2$ for $u\geq 0$. **Supposing** ψ_n to be N-functions, we denote by ψ_n^* the function complementary to ψ_n in the sense of Young. $(\psi_n^*)^{-1}$ will be the inverse to ψ_n^*. Then we write

$$(2)\qquad B_n = 2/(\psi_n^*)^{-1}(1).$$

Theorem 1. Let $\varphi = (\varphi_n)_{n=0}^{\infty}$ be a sequence of increasing φ-functions and let $\psi_n(u) = (\varphi_n^{-1}(u))^2$ be N-functions. Moreover, let $(B_n n^{-5\gamma/4})_{n=1}^{\infty}$ be a nonincreasing sequence and

(3) $$\sum_{n=1}^{\infty} B_n n^{-7\gamma/4} < \infty$$

with a $\gamma \in (0,2)$, B_n being given by (2). Let

(4) $$B_{2^{i+m}} \leq K B_{2^i} B_{2^m} \quad \text{and} \quad B_{2^i} \leq K B_{2^{i+1}}$$

for $i,m = 0,1,2,\ldots$, with a constant $K > 0$. Furthermore, let

(5) $$\psi_n(uv) \leq u^{\gamma} \psi_n(v) \qquad \text{for all} \quad u,v > 0.$$

Let us denote $H_i = \left\{ \sum_{k=2^i+1}^{2^{i+1}} k^{\frac{2}{\gamma} - \frac{3}{2}} B_k^{\frac{2}{\gamma}} c_k^2 \right\}^{\gamma/2}$. Then the condition $\sum_{i=0}^{\infty} H_i < \infty$ implies

$$\sum_{n=0}^{\infty} \varphi_n(\lambda |E_n^q(x) - E_{n-1}^q(x)|) < \infty \quad \text{a.e. for all } \lambda > 0.$$

Proof. Writing $f_n(x) = \varphi_n(\lambda |Z_n^q(x)|)$ for a fixed $\lambda > 0$, we have

(6) $$\int_0^1 \varphi_n(\lambda |Z_n^q(x)|)dx \leq 2 \|f_n\|_{\psi_n} \|1\|_{\psi_n^*} = B_n \|f_n\|_{\psi_n},$$

where $\|\ \|_{\psi_n}$ means the Luxemburg norm in the Orlicz space L^{ψ_n}.

However, we have

(7) $$\varrho_{\psi_n}(f_n) = \int_0^1 \psi_n(|f_n(x)|)dx = \frac{\lambda^2}{n^2} \sum_{k=0}^{n} (L_{n,k}^q)^2 k^2 c_k^2 .$$

By assumption (5), we have

(8) $$\frac{\lambda^2}{u^{2/\gamma}} (Z_n^q(x))^2 = \psi_n\left\{\varphi_n\left(\frac{1}{u^{1/\gamma}} \lambda |Z_n^q(x)|\right)\right\} \leq \psi_n\left\{\frac{1}{u} \varphi_n(\lambda |Z_n^q(x)|)\right\} = \psi_n\left(\frac{1}{u} f_n(x)\right) \quad \text{for every } u > 0.$$

Consequently,

(9) $$\|f_n\|_{\psi_n} \leq \lambda^{\gamma} \left(\int_0^1 (Z_n^q(x))^2 dx\right)^{\gamma/2} = \frac{\lambda^{\gamma}}{n^{\gamma}} \left(\sum_{k=0}^{n} (L_{n,k}^q)^2 k^2 c_k^2\right)^{\gamma/2} \leq \frac{\lambda^{\gamma}}{n^{5\gamma/4}} C_q^{\gamma/2} \left(\sum_{k=0}^{n} L_{n,k}^q k^2 c_k^2\right)^{\gamma/2}.$$

Hence, by (6), we have

(10) $$\sum_{n=2}^{\infty} \int_0^1 \varphi_n(\lambda |Z_n^q(x)|)dx \leq \lambda^{\gamma} C_q^{\gamma/2} \sum_{m=0}^{\infty} \sum_{n=2^m+1}^{2^{m+1}} \frac{B_n}{n^{5\gamma/4}} \left(\sum_{k=0}^{n} L_{n,k}^q k^2 c_k^2\right)^{\gamma/2}.$$

$$\sum_{n=2^m+1}^{2^{m+1}} \frac{B_n}{n^{5\gamma/4}} \left(\sum_{k=0}^{n} L^q_{n,k}\, k^2 c_k^{\,2}\right)^{\gamma/2} \le$$

$$\le \left\{\sum_{n=2^m+1}^{2^{m+1}} \left(\frac{B_n}{n^{5\gamma/4}}\right)^{\frac{2}{2-\gamma}}\right\}^{1-\frac{\gamma}{2}} \left\{\sum_{n=2^m+1}^{2^{m+1}} \sum_{k=0}^{n} L^q_{n,k}\, k^2 c_k^{\,2}\right\}^{\gamma/2} \le$$

$$\le 2^{\left(1-\frac{7\gamma}{4}\right)m} B_{2^m} \left\{\sum_{n=2^m+1}^{2^{m+1}} \sum_{k=0}^{n} L^q_{n,k}\, k^2 c_k^{\,2}\right\}^{\gamma/2} \le$$

$$\le 2^{\left(1-\frac{7\gamma}{4}\right)m} B_{2^m}\, 2 A_q^{\gamma/2} \left\{\sum_{i=0}^{m} \left(\sum_{k=2^i+1}^{2^{i+1}} k^2 c_k^{\,2}\right)^{\gamma/2} + |c_1|^{\gamma}\right\}.$$

(see [1], p.157). By (10), this gives

(11) $$\sum_{n=2}^{\infty} \int_0^1 \varphi_n\left(\lambda |Z_n^q(x)|\right) dx \le 2\lambda^{\gamma} \left(A_q C_q\right)^{\gamma/2} \left(S_1 + |c_1|^{\gamma} S_2\right),$$

where

$$S_1 = \sum_{m=0}^{\infty} 2^{\left(1-\frac{7\gamma}{4}\right)m} B_{2^m} \left(\sum_{i=0}^{m} \sum_{k=2^i+1}^{2^{i+1}} k^2 c_k^{\,2}\right)^{\gamma/2},$$

$$S_2 = \sum_{m=0}^{\infty} 2^{\left(1-\frac{7\gamma}{4}\right)m} B_{2^m} = \sum_{m=0}^{\infty} 2^{\left(1-\frac{1}{2}\gamma\right)m} B_{2^m} \left(2^m\right)^{-5\gamma/4}.$$

By assumption (3) and monotonity of the sequence $\left(B_n \cdot n^{-5\gamma/4}\right)_{n=1}^{\infty}$, we obtain $S_2 < \infty$. Moreover,

$$S_1 = \sum_{i=0}^{\infty} \left(\sum_{k=2^i+1}^{2^{i+1}} k^2\, c_k^{\,2}\right)^{\gamma/2} 2^{\left(1-\frac{7}{4}\gamma\right)i} B_{2^i} S_{3,i},$$

where

$$S_{3,i} = \sum_{m=0}^{\infty} 2^{\left(1-\frac{7}{4}\gamma\right)m} B_{2^{i+m}} \frac{1}{B_{2^i}} \le S_2 K,$$

by (4). Hence

$$S_1 \le S_2 K \sum_{i=0}^{\infty} \left(\sum_{k=2^i+1}^{2^{i+1}} k^2 c_k^{\,2}\right)^{\gamma/2} 2^{\left(1-\frac{7}{4}\gamma\right)i} B_{2^i}.$$

Supposing $2^i < k \le 2^{i+1}$ we obtain, by (4), $2^{\left(1-\frac{7}{4}\gamma\right)i} B_{2^i} \le 2^{5\gamma/4} K\, k^{1-\frac{7}{4}\gamma} B_k$.

Hence

$$S_1 \leq 2^{5\gamma/4} S_2 K^2 \sum_{i=0}^{\infty} H_i < \infty .$$

According to (11), this proves the theorem.

Remark 1. In particular, for $\varphi_n(u) = n^{\beta}|u|^{\gamma}$, β real, $0<\gamma<2$, $\psi_n(u) = n^{-2\beta/\gamma}|u|^{2/\gamma}$ is an N-function, and $B_n = B n^{\beta}$, where $B>0$ is a constant. Hence $\left(B_n n^{-5\gamma/4}\right)_{n=1}^{\infty}$ is nonincreasing for $\beta \leq \frac{5}{4}\gamma$ and (3) is equivalent to the inequality $\gamma > \frac{4}{7}(\beta+1)$; obviously, the second inequality implies the first one. Morever, (4) and (5) are satisfied. Thus, Theorem 1 implies the Theorem from [1] for $0<\gamma<2$.

Theorem 2. If the sequence $(c_n)_{n=0}^{\infty}$ is bounded, then Theorem 1 remains true for $0<\lambda<\lambda_0$ with a constant λ_0, if we replace the assumption (5) by the weaker one:

(12) $\varphi_n(uv) \leq u^{\gamma}\varphi_n(v)$ for $0<u\leq 1$ and $v>0$.

Proof. We estimate the norm $\|f_n\|_{\psi_n}$. Let $N_1 = \{n : \varrho_{\psi_n}(f_n) < 1\}$ and $N_2 = \{n : \varrho_{\psi_n}(f_n) \geq 1\}$.

Since (8) remains valid for $0<u\leq 1$, we have for $n \in N_1$ the inequality (9). It is sufficient to prove that N_2 is empty for small $\lambda > 0$. Let $n \in N_2$. Then, by (7),

$$\frac{\lambda^2}{n^2} \sum_{k=0}^{n} \left(L^q_{n,k}\right)^2 k^2 c_k^2 \geq 1$$

and, by $L^q_{n,k} \leq C_q n^{-1/2}$ and the assumption $|c_k| \leq M$ for $k=0,1,2,\ldots$, we get $1 \leq \frac{1}{6}\lambda^2 M^2 C_q^2 \frac{(n+1)(2n+1)}{n^2} \leq \lambda^2 M^2 C_q^2$. Taking $0<\lambda<\lambda_0 = \frac{1}{M C_q}$, we obtain a contradicion. Thus, N_2 is empty for such values λ.

Remark2. Let us note that the theses of Theorems 1 and 2 imply that the sequence $\left(Z_n^q(x)\right)_{n=0}^{\infty}$ belongs to the generalized Orlicz sequence space l^{φ} for a.e. $x \in \langle 0,1 \rangle$.

References

1. H.Musielak, A theorem concerning absolute Euler summability of orthogonal series, Functiones et Approximatio 14 (1984), 155-158.
2. V.N.Spevakov, Absolute summability of orthenormal series by E,q - methods, Izv.Vys.Uc.Uav., 1981. N$^{\circ}$ 2(225), 70-73.

3. O.A.Ziza, On summability of orthonormal series by Euler methods, Matem.Sb., 1965, 66(108), 3, 354-377.

Institute of Mathematics
A.Mickiewicz University
Matejki 48/49
60-769 Poznań Poland