CONSTRUCTIVE THEORY OF FUNCTIONS'87

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ON GENERALIZED ORLICZ SEQUENCE SPACES OF FOURIER COEFFICIENTS FOR TRIGONOMETRIC GAP SERIES. II.

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1. Let

$$\sum_{k=1}^{\infty} \left(a_k(f)\cos n_k x + b_k(f)\sin n_k x \right)$$

be the Fourier series of a function $f \in L^p_{2\pi}$, $1 < p \leq 2$, with respect to the trigonometric gap system $\cos n_1 x$, $\sin n_1 x$, $\cos n_2 x$, $\sin n_2 x$, ... in $[0,2\pi]$, where (n_k) is an increasing sequence of indices. We shall write $A_\nu = \{k \in \mathbb{N} : 2^{\nu-1}\pi \leq n_k < 2^\nu \pi\}$, $\nu = 1,2,\ldots$, and we denote by k_o the least integer in A_1. We observe that taking an increasing function $l(x)$, $x \geqslant 0$, such that $l(k) = n_k$ and denoting by l^{-1} the inverse function of l, the number $|A_\nu|$ of elements of A_ν satisfies the inequality $|A_\nu| < [l^{-1}(2^\nu\pi) - l^{-1}(2^{\nu-1}\pi)] + 1 = N_\nu$ for $\nu = 1,2,\ldots$. Now, we associate with every function $f \in L^p_{2\pi}$, $1<p\leq 2$, the sequence $c(f) = a_{k_o}(f)$, $b_{k_o}(f)$, $a_{k_o+1}(f)$, $b_{k_o+1}(f)$, ... of Fourier coefficients of f with respect to the above trigonometric gap system. There will be investigated the operator $c: f \longrightarrow c(f)$ as an operator from a certain subspace of $L^p_{2\pi}$ to the generalized Orlicz sequence space l^φ generated by a sequence $\varphi = (\varphi_n)$ of φ-functions φ_n (for terminology, see [3]). This was done in case when this space was constructed by means of integral modulus of continuity of f in $L^p_{2\pi}$, in [4]; the results were an extension of the well-known Berstein's theorem on absolute convergence of Fourier series and its generalizations ([7], Chapter VI, (3.1)). Now, generalizing the results of [1] to gap series, we do the same in case when the subspace of $L^p_{2\pi}$ is a two-modular space constructed by usual modulus of continuity of f in $C_{2\pi}$ and by the Φ-variation of f; this extends the well-known Zygmund's theorem on absolute convergence of Fourier series ([7], Chapter VI, (3.6)). We shall need the following assumptions on $\varphi = (\varphi_n)$ (see [4]):

A.1. there exists a constant $C_1 \geqslant 1$ and a sequence of integers $(m(\nu))$ with $m(\nu)\in A_\nu$ such that $\varphi_n(u)\le C_1\varphi_{m(\nu)}(u)$ for $u\geqslant 0$ and $n\in A_\nu$;

A.2. the functions $\bar{\varphi}_n(u)=\varphi_n(u^{1/q})$, $u\geqslant 0$, $\frac{1}{p}+\frac{1}{q}=1$, are concave.

Let us remark that A.2 implies the inequality

(1) $$\varphi_n(2u)\leqslant 2^{1/q}\varphi_n(u) \text{ for } u\geqslant 0,\ n=1,2,\ldots .$$

We shall need still the following assumption :

A.3. There exist a φ-function ϕ , a nondecreasing function $\Psi:\mathbb{R}_+\longrightarrow\mathbb{R}_+$ and a constant $C_2>0$ such that

$$|u|^p\leqslant C_2\,\phi(u)\Psi(u) \quad\text{for } u\geqslant 0 ;$$

we may suppose Ψ to be right-continuous at 0.

2. We denote by $C_{2\pi}V_\phi$ the space of continuous, 2π-periodic functions f of bounded ϕ-variation $\overset{2\pi}{\underset{0\phi}{V}}(f)$ in sense of L.C.Young [6] in $[0,2\pi]$. Writing $\omega(f,\delta)$ for the modulus of continuity of such a function f, we have the following

Lemma. If $f\in C_{2\pi}V_\phi$ and A.3 is satisfied, then

$$\sum_{k\in A_\nu}\left(|a_k(f)|^q+|b_k(f)|^q\right)\leqslant 2^{-q/2}\,5^{q/p}\,C_2^{\,q/p}\,\Psi^{q/p}\!\left(\omega(f,\tfrac{\pi}{2^\nu})\right)2^{-\nu q/p}\times$$
$$\times\left(\overset{2\pi}{\underset{0}{V}}{}_\phi(f)\right)^{q/p}\quad\text{for } \nu=1,2,\ldots,\ 1<p\leqslant 2,\ \frac{1}{p}+\frac{1}{q}=1 .$$

Proof. Writing $F_h(x)=f(x+h)-f(x-h)$ and denoting $S_\nu=\sum_{k\in A_\nu}\left(|a_k(f)|^q+|b_k(f)|^q\right)$, we have by [4], formula (**) ,

$$S_\nu^{1/q}\leqslant\frac{1}{\sqrt{2}}\left\{\frac{1}{\pi}\int_0^{2\pi}|F_{2^{-\nu-1}}(x)|^p\,dx\right\}^{1/p} .$$

Arguing as in [1], 2.2, we obtain easily

$$S_\nu^{1/q}\leqslant\frac{C_2^{1/p}}{\sqrt{2}}\,\Psi^{1/p}\!\left(\omega(f,\tfrac{\pi}{2^\nu})\right)2^{-\nu/p}\left(\overset{4\pi}{\underset{-2\pi}{V}}{}_\phi(f)\right)^{1/p}\leqslant$$
$$\leqslant\frac{1}{\sqrt{2}}\,5^{1/p}\,C_2^{1/p}\,\Psi^{1/p}\left(\omega(f,\tfrac{\pi}{2^\nu})\right)2^{-\nu/p}\left(\overset{2\pi}{\underset{0\phi}{V}}(f)\right)^{1/p} ,$$

which proves the Lemma.

Now, let us write

$$\varrho(c)=\sum_{n=k_0}^{\infty}\varphi_n(|c_n|)\quad\text{for } c=(c_n) ,$$

$$\varrho_{\nu,\varphi}(f)=2\,C_1\,N_\nu\,\varphi_{m(\nu)}\Big\{2^{-3/2}(10\,C_2)^{1/p}\Big(\overset{2\pi}{\underset{0\phi}{V}}(f)\Big)^{1/p}N_\nu^{-1/q}\times$$
$$\times\Psi^{1/p}\left(\omega(f,\tfrac{\pi}{2^\nu})\right)\Big\} ,\qquad \varrho_\varphi(f)=\sum_{\nu=1}^{\infty}\varrho_{\nu,\varphi}(f) .$$

Then there holds the following

Theorem 1. Let $\varphi=(\varphi_n)$ satisfy A.1 and A.2, and A.3 be satisfied. Then for every $f\in C_{2\pi}V_\Phi$, $1<p\leq 2$, $\frac{1}{p}+\frac{1}{q}=1$, there holds the inequality

$$(2)\qquad \varrho(c(f))\leq \varrho_\varphi(f)\ .$$

Proof. By the assumptions A.1, A.2 and by Lemma, we obtain easily

$$\sum_{k\in A_\nu}\left(\varphi_k(|a_k(f)|+\varphi_k(|b_k(f)|)\right)\leq$$

$$\leq 2\,c_1|A_\nu|\varphi_{m(\nu)}\left\{\frac{1}{2|A_\nu|}2^{-q/2}\,5^{q/p}\,c_2^{q/p}\psi^{q/p}\left(\omega(f,\tfrac{\pi}{2\nu})\right)\left(\overset{2\pi}{\underset{0}{V}}(f)\right)^{q/p}\right\}\leq\varrho_{\nu,\varphi}(f)\ .$$

Hence

$$\varrho(c(f))=\sum_{\nu=1}^{\infty}\sum_{k\in A_\nu}\left(\varphi_k(|a_k(f)|)+\varphi_k(|b_k(f)|)\right)\leq\varrho_\varphi(f)\ .$$

This proves Theorem 1.

Now, let us specialize $\varphi_n(u)=n^\beta|u|^\gamma$ with any real β and $0<\gamma\leq q$. We obtain

Corollary 1. Let $0<\gamma\leq q$, β real, $f\in C_{2\pi}V_\Phi$, with A.3. If

$$\sum_{\nu=1}^{\infty}(m(\nu))^\beta N_\nu^{-\gamma/q}\,\psi^{\gamma/p}\left(\omega(f,\tfrac{\pi}{2\nu})\right)<\infty\ ,$$

then

$$\sum_{n=1}^{\infty}n^\beta\left(|a_n(f)|^\gamma+|b_n(f)|^\gamma\right)<\infty\ .$$

In particular, if $\Phi(u)=|u|^r$ with $1\leq r\leq p$ and the sequence (n_k) satisfies gap conditions $k^s=O(n_k)$ for an $s>0$ or $n_{k+1}/n_k\geq\alpha>1$, Corollary 1 generalizes known results on Fourier series (see e.g[7], Chapter VI, § 3, also [1],p.149, Theorem 3.2).

3. We shall apply Theorem 1 in order to obtain a result on continuity of the linear operator $c\colon f\to c(f)$. We define in $C_{2\pi}V_\Phi$ the following two pseudomodulars :

$$\varrho_\Phi^{(1)}(f)=\left(\overset{2\pi}{\underset{0}{V}}_\Phi(f)\right)^{1/p}\ ,$$

$$\varrho_\varphi^{(2)}(f)=\sum_{\nu=1}^{\infty}N_\nu\,\varphi_{m(\nu)}\left\{N_\nu^{-1/q}\,\psi^{1/p}\left(\omega(f,\tfrac{\pi}{2\nu})\right)\right\}\ ,$$

Let $(C_{2\pi}V_\Phi)_\varphi=\{f\in C_{2\pi}V_\Phi:\ \varrho_\varphi^{(2)}(\lambda f)\to 0\text{ as }\lambda\to 0_+\}$.

Theorem 2. Under the assumptions A.1, A.2 and A.3, $c\colon f\to c(f)$ is a linear operator continuous from $(C_{2\pi}V_\Phi)_\varphi$ provided with two-modular convergence $\langle(C_{2\pi}V_\Phi)_\varphi,\varrho_\varphi^{(1)},\varrho_\varphi^{(2)}\rangle$ to the generalized Orlicz sequence space l^φ.

Proof. Two-modular convergence to 0 of a sequence (f_n), $f_n\in(C_{2\pi}V_\Phi)_\varphi$, means that f_n is $\varrho_\Phi^{(1)}$-bounded and $\varrho_\Phi^{(2)}$-convergent to 0 (see[5]). The first one implies $\varrho_\Phi^{(1)}(\alpha f_n)\leq M$ for $n=1,2,\ldots$ with some constants α, $M>0$ (see[3] , Theorem 5.5). Applying (2) to αf_n in place of f, we obtain

$$\varrho(\alpha c(f_n))\leq\varrho_\varphi(\alpha f_n)\leq$$

$$\leq 2 C_1 \sum_{\nu=1}^{\infty} N_\nu \varphi_{m(\nu)}\left\{2^N N_\nu^{-1/q} \psi^{1/p}\left(\omega\left(\alpha f_n, \frac{\pi}{2^\nu}\right)\right)\right\},$$

where N is the least nonnegative integer such that $2^{-3/2}(10\, C_2)^{1/p} M \leq 2^N$. By (1), we thus obtain $\varrho(\alpha c(f_n)) \leq 2\, C_1\, 2^{N/q} \varrho_\varphi^{(2)}(\alpha f_n)$ for sufficiently small $\alpha > 0$. But $\varrho_\varphi^{(2)}(\alpha f_n) \to 0$ as $n \to \infty$ for small $\alpha > 0$, and so $\varrho(\alpha c(f_n)) \to 0$ as $n \to \infty$, too. Let us still remark that due to (1), modular convergence and norm convergence in l^φ are equivalent, so we do not need to distinguish between them.

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