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ON IMBEDDINGS OF WEIGHTED SOBOLEV SPACES

Bohumír Opic, Petr Gurka

1. Introduction

The paper deals with imbeddings of weighted Sobolev spaces into weighted Lebesgue spaces and its aim is to establish some conditions on weight functions v_0, v_1, w which guarantee the validity of the continuous imbedding

(1.1) $$W^{1,p}(\Omega;v_0,v_1) \hookrightarrow L^q(\Omega;w)$$

or the compact imbedding

(1.2) $$W^{1,p}(\Omega;v_0,v_1) \hookrightarrow\hookrightarrow L^q(\Omega;w) .$$

Similar problems have been studied by various authors. In [5] P. I. LIZORKIN and M. OTELBAEV gave some necessary and sufficient conditions under which the above imbeddings hold with Ω a bounded domain in $\mathbb{R}^N$ and $1 < p \leq q < \infty$. Unfortunately their conditions are rather complicated. Similar conditions were established by N.K. KORENEV [3] who studied the imbedding

(1.3) $$W_0^{1,p}(\Omega;v_0,v_1) \hookrightarrow\hookrightarrow L^q(\Omega;w) .$$

W. ZAJACZKOWSKI [11] investigated the imbedding (1.2) in the case $1 < p \leq q < \infty$ for power-type weights. Other results for an unbounded domain Ω in $\mathbb{R}^N$ and $p = q$ were given by B. OPIC in [6], [7] and by B. OPIC and P. GURKA in [9] (here $q^{-1} = p^{-1} - (Nr)^{-1}$, $1 < r < p < Nr$). General sufficient and necessary conditions for the weight functions v_0, v_1, w under which the imbedding (1.2) takes place were obtained by A. AVANTAGGIATI in [1] and by B. OPIC in [8]. Further, the problem of the continuous imbedding (1.1) (with $p = q$ and with a bounded domain Ω) was studied by A. KUFNER in the book [3].

Throughout this paper we will suppose that Ω is a bounded domain in $\mathbb{R}^N$ with a boundary $\partial\Omega$. If $x \in \Omega$ then we set $d(x) = \mathrm{dist}(x,\partial\Omega)$. By $W(\Omega)$ we denote the set of *weight functions* on Ω, i.e., the set of all measurable, a.e. in Ω positive and finite functions.

For $w \in W(\Omega)$, $1 \leq q < \infty$ the *weighted Lebesgue space* $L^q(\Omega;w)$ is the set of all measurable functions u defined on Ω with a finite norm

$$\|u\|_{q,\Omega,w} = \left(\int_\Omega |u(x)|^q \, w(x) \, dx\right)^{1/q} .$$

Throughout the paper we assume that

$$v_0, v_1 \in W(\Omega) \cap L^1_{loc}(\Omega) , \quad v_0^{-1/p}, v_1^{-1/p} \in L^{p'}_{loc}(\Omega) \quad (p' = \frac{p}{p-1}).$$

We define the *weighted Sobolev space* $W^{1,p}(\Omega;v_0,v_1)$ as the set of all functions $u \in L^p(\Omega;v_0)$ which have on Ω distributional derivatives $\frac{\partial u}{\partial x_i} \in L^p(\Omega;v_1)$, $i = 1,2,\dots,N$. The space $W^{1,p}(\Omega;v_0,v_1)$ with the norm

$$\|u\|_{1,p,\Omega,v_0,v_1} = \left(\|u\|^p_{p,\Omega,v_0} + \sum_{i=1}^N \left\|\frac{\partial u}{\partial x_i}\right\|^p_{p,\Omega,v_1}\right)^{1/p}$$

is a Banach space. Further, we define

$$W_0^{1,p}(\Omega;v_0,v_1) = \overline{C_0^\infty(\Omega)}^{\|\cdot\|_{1,p,\Omega,v_0,v_1}} .$$

Given $x \in \mathbb{R}^N$ and $R > 0$, we put $B(x,R) = \{y \in \mathbb{R}^N;\ |x - y| < R\}$. By $C^{0,\kappa}$, $0 < \kappa \leq 1$, we denote the class of all bounded domains in $\mathbb{R}^N$ with a κ-Hölder boundary (in the sense of [4], Definition 4.3).

2. Continuous and compact imbeddings - general weights

In this section we present the main results of our paper. We will consider the case when the weight functions may have singularities or degenerations only on the boundary $\partial\Omega$ of the bounded doamin Ω. (That is, on any bounded domain G, $\overline{G} \subset \Omega$, the weight functions are bounded from above and from below by positive constants, and thus we can use the fact that the classical Sobolev imbedding theorems hold on G.)

Throughout this section we will suppose

2.1. Assumptions. (i) Let $\{\Omega_n\}_{n=1}^\infty$ be a sequence of domains such that

$$\Omega_n \in C^{0,1} , \quad \{x \in \Omega;\ n^{-1} < d(x)\} \subset \Omega_n \subset \{x \in \Omega;\ (n+1)^{-1} < d(x)\} . \quad ^{*)}$$

Put $\Omega^n = \operatorname{int}(\Omega \setminus \Omega_n)$.

(ii) There exist $n_0 \in \mathbb{N}$, $n_0 \geq 3$, a positive function r defined on Ω^{n_0} and a constant $c_r \geq 1$ such that

$$r(x) \leq d(x)/3 , \quad x \in \Omega^{n_0} ;$$

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*) Evidently $\Omega_n \subset \Omega_{n+1} \subsetneq \Omega$, $\Omega = \bigcup_{n=1}^{\infty} \Omega_n$.

$$x \in \Omega^{n_0}, \quad y \in B(x, r(x)) \Rightarrow c_r^{-1} \le \frac{r(y)}{r(x)} \le c_r .$$

The theorems in this section are based on the following two lemmas, the proofs of which can be found in [8].

2.2. Lemma. Let $p, q \in \langle 1, \infty)$. Suppose that

$$W^{1,p}(\Omega_n; v_0, v_1) \hookrightarrow L^q(\Omega_n; w), \quad n \ge n_0$$

and

$$\lim_{n\to\infty} \sup_{\|u\|_{1,p,\Omega,v_0,v_1} \le 1} \|u\|_{q,\Omega^n,w} < \infty . \tag{2.2}$$

Then

$$W^{1,p}(\Omega; v_0, v_1) \hookrightarrow L^q(\Omega; w) . \tag{2.3}$$

Conversely, if the imbedding (2.3) holds then the condition (2.2) is satisfied.

2.3. Lemma. Let $p, q \in \langle 1, \infty)$. Suppose that

$$W^{1,p}(\Omega_n; v_0, v_1) \hookrightarrow\hookrightarrow L^q(\Omega_n; w), \quad n \ge n_0$$

and

$$\lim_{n\to\infty} \sup_{\|u\|_{1,p,\Omega,v_0,v_1} \le 1} \|u\|_{q,\Omega^n,w} = 0 . \tag{2.4}$$

Then

$$W^{1,p}(\Omega; v_0, v_1) \hookrightarrow\hookrightarrow L^q(\Omega; w) . \tag{2.5}$$

Conversely, if the imbedding (2.5) holds then the condition (2.4) is satisfied.

Now, we are going to present our main results.

2.4. Theorem (the continuous imbedding). Let $1 \le p \le q < \infty$, $N^{-1} \ge p^{-1} - q^{-1}$ and let the following conditions be fulfilled :

C1. $$W^{1,p}(\Omega_n; v_0, v_1) \hookrightarrow L^q(\Omega_n; w), \quad n \ge n_0 .$$

C2. There exist positive constants $c_0 \le C_0$, $c_1 \le C_1$ and positive measurable functions a_0, a_1 defined on Ω^{n_0} such that for every $x \in \Omega^{n_0}$ and for a.e. $y \in B(x, r(x))$,

$$c_0\, a_0(x) \le w(y) \le C_0\, a_0(x) ,$$

$$c_1\, a_1(x) \le v_1(y) \le C_1\, a_1(x) .$$

C3. There exist positive constants $k \le K$ such that

$$k\, v_0(x) \le v_1(x)\, r^{-p}(x) \le K\, v_0(x) \quad \text{for a.e. } x \in \Omega^{n_0} .$$

Then

$$W^{1,p}(\Omega; v_0, v_1) \hookrightarrow L^q(\Omega; w)$$

if and only if

$$\text{C4.}\qquad \lim_{n\to\infty}\ \sup_{x\in\Omega^n} \frac{a_0^{1/q}(x)}{a_1^{1/p}(x)}\, r^{\frac{N}{q}-\frac{N}{p}+1}(x) < \infty .$$

2.5. Theorem (the compact imbedding). Let us suppose $1 \le p \le q < \infty$, $N^{-1} > p^{-1} - q^{-1}$ and the conditions C2, C3. Moreover, let

$$\text{C1*.}\qquad W^{1,p}(\Omega_n;v_0,v_1) \hookrightarrow\hookrightarrow L^q(\Omega_n;w) , \quad n \ge n_0 .$$

Then

$$W^{1,p}(\Omega;v_0,v_1) \hookrightarrow\hookrightarrow L^q(\Omega;w)$$

if and only if

$$\text{C4*.}\qquad \lim_{n\to\infty}\ \sup_{x\in\Omega^n} \frac{a_0^{1/q}(x)}{a_1^{1/p}(x)}\, r^{\frac{N}{q}-\frac{N}{p}+1}(x) = 0 .$$

For the proof of Theorems 2.4 and 2.5 see [2].

From Theorems 2.4 and 2.5 we obtain :

2.6. Example. Let $1 \le p \le q < \infty$ and let α , β be real numbers.

(i) Suppose $N^{-1} \ge p^{-1} - q^{-1}$. Then

$$W^{1,p}(\Omega;d^{\beta-p},d^{\beta}) \hookrightarrow L^q(\Omega;d^{\alpha})$$

if and only if

$$N(\frac{1}{q} - \frac{1}{p}) + \frac{\alpha}{q} - \frac{\beta}{p} + 1 \ge 0 .$$

(ii) Suppose $N^{-1} > p^{-1} - q^{-1}$. Then

$$W^{1,p}(\Omega;d^{\beta-p},d^{\beta}) \hookrightarrow\hookrightarrow L^q(\Omega;d^{\alpha})$$

if and only if

$$N(\frac{1}{q} - \frac{1}{p}) + \frac{\alpha}{q} - \frac{\beta}{p} + 1 > 0 .$$

2.7. Example. Let $1 \le p \le q < \infty$ and let α , β be real numbers. For $x \in \Omega$ we put

$$w(x) = \exp\left(\alpha/d(x)\right) ,$$

$$v_0(x) = d^{-2p}(x) \exp\left(\beta/d(x)\right) , \quad v_1(x) = \exp\left(\beta/d(x)\right) .$$

(i) Suppose $N^{-1} \ge p^{-1} - q^{-1}$. Then

$$W^{1,p}(\Omega;v_0,v_1) \hookrightarrow L^q(\Omega;w)$$

if and only if

$$\frac{\alpha}{q} - \frac{\beta}{p} \le 0 .$$

(ii) Suppose $N^{-1} > p^{-1} - q^{-1}$. Then

$$W^{1,p}(\Omega;v_0,v_1) \hookrightarrow\hookrightarrow L^q(\Omega;w)$$

if and only if

$$\frac{\alpha}{q} - \frac{\beta}{p} \leq 0 .$$

3. The power-type weights

Applying the results of Example 2.6 and the imbedding theorems concerning the imbedding

$$W^{1,p}(\Omega;d^{\varepsilon},d^{\varepsilon}) \hookrightarrow L^p(\Omega;d^{\eta})$$

(see [4]) we can derive the following two assertions :

3.1. Theorem. Let $\Omega \in C^{0,\kappa}$, $0 < \kappa \leq 1$, $1 \leq p \leq q < \infty$, $N^{-1} \geq p^{-1} - q^{-1}$. Then

$$W^{1,p}(\Omega;d^{\beta},d^{\beta}) \hookrightarrow L^q(\Omega;d^{\alpha})$$

if

$$\beta > \kappa p \quad , \quad N(\frac{1}{q} - \frac{1}{p}) + \frac{\alpha}{q} - \frac{\beta}{\kappa p} + 1 \geq 0$$

or

$$\kappa p \geq \beta > \kappa(p-1) \; , \quad N(\frac{1}{q} - \frac{1}{p}) + \frac{\alpha}{q} - \frac{\beta}{p} + \kappa \geq 0$$

or

$$\kappa(p-1) \geq \beta \quad , \quad N(\frac{1}{q} - \frac{1}{p}) + \frac{\alpha}{q} - \frac{\kappa(p-1)}{p} + \kappa > 0 .$$

3.2. Theorem. Let $\Omega \in C^{0,\kappa}$, $0 < \kappa \leq 1$, $1 \leq p \leq q < \infty$, $N^{-1} > p^{-1} - q^{-1}$. Then

$$W^{1,p}(\Omega;d^{\beta},d^{\beta}) \hookrightarrow\hookrightarrow L^q(\Omega;d^{\alpha}) .$$

if

$$\beta > \kappa p \quad , \quad N(\frac{1}{q} - \frac{1}{p}) + \frac{\alpha}{q} - \frac{\beta}{\kappa p} + 1 > 0$$

or

$$\kappa p \geq \beta > \kappa(p-1) \; , \quad N(\frac{1}{q} - \frac{1}{p}) + \frac{\alpha}{q} - \frac{\beta}{p} + \kappa > 0$$

or

$$\kappa(p-1) \geq \beta \quad , \quad N(\frac{1}{q} - \frac{1}{p}) + \frac{\alpha}{q} - \frac{\kappa(p-1)}{p} + \kappa > 0 .$$

Now, let us briefly mention the case $1 \leq q < p < \infty$. Here the principal role is played by the Hardy inequality

$$\left(\int_0^b |u(t)|^q \, t^{\varepsilon} \, dt\right)^{1/q} \leq c \left(\int_0^b |u'(t)|^p \, t^{\eta} \, dt\right)^{1/p} \tag{3.1}$$

($0 < b < \infty$, $c > 0$ is a constant independent of u) which holds for all functions u satisfying the conditions

$$u \in AC((0,b>) \ , \quad u(b) = 0$$

or

$$u \in AC(<0,b)) \ , \quad u(0) = 0$$

if and only if

(3.2) $\eta \le p - 1 \ , \quad \varepsilon > -1 \quad \vee \quad \eta > p-1 \ , \quad \varepsilon > \eta\frac{q}{p} - \frac{q}{p'} - 1$

or

(3.3) $\eta < p - 1 \ , \quad \varepsilon > \eta\frac{q}{p} - \frac{q}{p'} - 1 \ ,$

respectively.

With help of the inequality (3.1), the method of local coordinates, and Lemmas 2.2 and 2.3 it is possible to prove the following imbedding theorem. Let us remark that the imbedding in this theorem is compact in view of the strictness of the inequalities for ε in (3.2).

3.3. Theorem. Let $\Omega \in C^{0,\kappa}$, $0 < \kappa \le 1$, $1 \le q < p < \infty$. Then

$$W^{1,p}(\Omega;d^{\beta},d^{\beta}) \subsetneq\subsetneq L^{q}(\Omega;d^{\alpha})$$

if

(3.4) $\kappa(p - 1) + \kappa\frac{p}{q} < \beta \qquad , \quad (\frac{1}{q} - \frac{1}{p}) + \frac{\alpha}{q} - \frac{\beta}{\kappa p} + 1 > 0$

or

(3.5) $\kappa(p - 1) < \beta \le \kappa(p - 1) + \kappa\frac{p}{q} \quad , \quad \kappa(\frac{1}{q} - \frac{1}{p}) + \frac{\alpha}{q} - \frac{\beta}{p} + \kappa > 0$

or

(3.6) $\beta \le \kappa(p - 1) \ , \quad \alpha > -\kappa \ .$ *)

3.4. Remark. (i) Analogous results (as in Theorems 3.1 - 3.3) can be obtained for the imbeddings

$$W_0^{1,p}(\Omega;d^{\beta},d^{\beta}) \subsetneq L^{q}(\Omega;d^{\alpha})$$

and

$$W_0^{1,p}(\Omega;d^{\beta},d^{\beta}) \subsetneq\subsetneq L^{q}(\Omega;d^{\alpha})$$

(see [10]).

(ii) If $\Omega \in C^{0,1}$ then in some cases it is possible to give necessary and sufficient conditions for the above imbeddings (see [10]).

*) The inequality $\alpha > -\kappa$ can be written as

$$\frac{\alpha}{q} - \frac{\kappa(p - 1)}{p} + \kappa(\frac{1}{q} - \frac{1}{p}) + \kappa > 0 \ ,$$

which is the second inequality from (3.5) with $\beta = \kappa(p - 1)$.

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