CONSTRUCTIVE THEORY OF FUNCTIONS'87

SOFIA, 1988

SOME ELEMENTARY INEQUALITIES IN CONNECTION WITH X^p-SPACES

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0. Introduction. Let (Ω,Σ,μ) be a σ-finite measure space and let $L^0(\Omega)$ denote the space of μ-measurable functions defined and finite a.e. on Ω. A Banach subspace $X = (X,\|\cdot\|_X)$ of $L^0(\Omega)$ is a Banach lattice (on (Ω,Σ,μ)) if, for every $x \in X$, the following implication holds:

$$y \in L^0(\Omega)\ ,\ |y|\le|x|\ \mu\text{-a.e.} \Rightarrow y \in X \text{ and } \|y\|_X \le \|x\|_X . \tag{0.1}$$

The space X^p, $-\infty < p < \infty$, $p \neq 0$, consists of all $x \in L^0(\Omega)$, satisfying

$$\|x\|_{X^p} = (\||x|^p\|_X)^{1/p} < \infty .$$

For the case $p < 0$ we assume that $x = x(t) \neq 0$ for all $t \in \Omega$.

The X^p-spaces have been used and studied in some recent papers (see e.g [3], [11] and [13]). In this paper we give some additional information about the X^p-spaces. In particular we prove that suitable versions of some classical inequalities (e.g. those by Hölder, Minkowski, Clarkson, Beckenbach) hold in this more general situation. We point out several applications and connections to other results of this kind.

1. On the X^p-spaces. Let X be a complete subspace of $L^0(\Omega)$ satisfying (0.1). If X instead of the usual triangle inequality only satisfies the c-triangle inequality $(c>1)$

$$\|x+y\|_X \le c(\|x\|_X + \|y\|_X) ,$$

then we say that X is a c-quasi-Banach lattice. First we state the following useful theorem.

Theorem 1.1 *Let X be a Banach lattice. Then the space X^p is a Banach lattice if $p \ge 1$ and a c-quasi-Banach lattice with $c = 2^{(1-p)/p}$ if $0 < p < 1$.*

Proof A proof in the case when $p \ge 1$ can be found in [11] (see also our Theorem 4.1). Let $0 < p < 1$. First we note that if $a, b \ge 0$, then

$$(a+b)^p \le a^p + b^p \le 2^{1-p}(a+b)^p . \tag{1.1}$$

We use (0.1), the left hand inequality of (1.1) and the triangle inequality and find that

$$\| |x+y|^p \|_X \le \| |x|^p + |y|^p \|_X \le \| |x|^p \|_X + \| |y|^p \|_X .$$

Therefore, according to the right hand inequality of (1.1), we get

$$\|x+y\|_{X^p} \le (\|x\|^p_{X^p} + \|y\|^p_{X^p})^{1/p} \le 2^{(1-p)/p}(\|x\|_{X^p} + \|y\|_{X^p}) .$$

We conclude that X^p satisfies the $2^{(1-p)/p}$-triangle inequality. Therefore there exists a norm $\|\cdot\|^*_{X^p}$ on X^p such that

$$\|x\|^*_{X^p} \le \|x\|^p_{X^p} \le 2\|x\|^*_{X^p} , \tag{1.2}$$

see [14] or [6,p.59]. Now let $x_i \in X^p$, $i = 1, 2, \ldots$, and assume that $\Sigma_1^\infty \|x_i\|^*_{X^p} < \infty$. Then, by (1.2),

$$\sum_1^\infty \|x_i\|^p_{X^p} = \sum_1^\infty \| |x_i|^p \|_X < \infty$$

and we conclude that $\Sigma_1^\infty |x_i|^p \in X$. We put $x = \Sigma_1^\infty x_i$ and note that $|x|^p \le \Sigma_1^\infty |x_i|^p$. Therefore the lattice property (0.1) implies that $|x|^p \in X$. Hence $x \in (X^p, \|\cdot\|^*_{X^p})$ and we have proved that X^p is complete. Since it is obvious that the lattice property (0.1) holds for the space X^p, $p > 0$, the theorem follows.

<u>Remark 1.1</u>. The spaces X^p and $X^{1/p}$, $1 < p < \infty$, are sometimes called the p-convexification and the p-concavification of X, respectively. Concerning this terminology see [13] and compare with our Theorem 6.1.

In this connection we also raise the following

<u>open question</u>: Let X be a quasi-Banach lattice. Does there exist an equivalent quasi-norm such that X^p is a Banach lattice for some $p \ge 1$? (Compare with [13,p.144] and [11].)

<u>2. Hölder's inequality</u>. In the sequel we let X and N denote a Banach space and a positive integer, respectively.

<u>Theorem 2.1</u>. <u>Let</u> $x_i \in X^{p_i}$, $0 < p_i < \infty$, $i = 1, 2, \ldots, N$. <u>Then</u> $\pi_1^N x_i \in X^r$, <u>where</u> $\frac{1}{r} = \Sigma_1^N \frac{1}{p_i}$, <u>and</u>

$$\|\prod_1^N x_i\|_{X^r} \le \prod_1^N \|x_i\|_{X^{p_i}} . \tag{2.1}$$

<u>Proof</u> We put $y_i = x_i/\|x_i\|_{X^{p_i}}$, $i = 1, 2, \ldots, N$, and $\Omega_o = \{t \in \Omega : \pi_1^N y_i(t) \ne 0\}$. We use the fact that $f(u) = \exp(u)$ is a convex function and find that, for every $t \in \Omega_o$,

$$\prod_1^N |y_i(t)|^r = \exp(r \ln \prod_1^N |y_i(t)|) = \exp(\sum_1^N \frac{r}{p_i} \ln |y_i(t)|^{p_i}) \le \sum_1^N \frac{r}{p_i} |y_i(t)|^{p_i} .$$

Therefore

$$\|\prod_1^N y_i\|^r_{X^r} \le \sum_1^N \frac{r}{p_i} \|y_i\|^{p_i}_{X^{p_i}} = \sum_1^N \frac{r}{p_i} = 1$$

and (2.1) follows.

Remark 2.1. For the case $N = 2$ see [11]. We also remark that equality in (2.1) occurs if

$$c_1|x_1|^{p_1} = c_2|x_2|^{p_2} = \dots = c_N|x_N|^{p_N}$$

for some positive constants c_i, $i = 1, 2, \dots, N$. (The notation $x = y$ means that $x(t) = y(t)$ for almost all $t \in \Omega$.) –

3. Symmetric versions of Hölder's inequality. In this section we generalize the main results obtained in [1].

Theorem 3.1. Let p, q and r be non-zero real numbers satisfying $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$. Then

a) $\|xy\|_{X^r} \le \|x\|_{X^p}\|y\|_{X^q}$,

if $p > 0$, $q > 0$ and $r > 0$ or if $p < 0$, $q > 0$ and $r < 0$ or if $p > 0$, $q < 0$ and $r < 0$.

b) $\|xy\|_{X^r} \ge \|x\|_{X^p}\|y\|_{X^q}$,

if $p > 0$, $q < 0$ and $r > 0$ or if $p < 0$, $q > 0$ and $r > 0$ or if $p < 0$, $q < 0$ and $r < 0$.

Proof Let $p > 0$, $q > 0$ and $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$. Then, according to Theorem 2.1,

$$\|xy\|_{X^r} \le \|x\|_{X^p}\|y\|_{X^q} . \tag{3.1}$$

Let $p > 0$, $q < 0$ and $r > 0$. We use (3.1) in the following way with p, q and r replaced by r/p, $-q/p$ and 1, respectively:

$$\|x\|^p_{X^p} = \| |xy|^p |y|^{-p} \|_X \le \| |xy|^p \|_{X^{r/p}} \| |y|^{-p} \|_{X^{-q/p}} = \|xy\|^p_{X^r} \|y\|^{-p}_{X^q} .$$

Hence

$$\|xy\|_{X^r} \ge \|x\|_{X^p}\|y\|_{X^q} . \tag{3.2}$$

In the same way we find that (3.2) also holds for the symmetric case $p < 0$, $q > 0$ and $r > 0$.

Let $p < 0$, $q > 0$ and $r < 0$ or $p > 0$, $q < 0$ and $r < 0$. We use (3.2) in the following way with p, q and r replaced by $-p$, $-q$ and $-r$, respectively:

$$\|xy\|_{X^r} = \| |xy|^{-1}\|^{-1}_{X^{-r}} \le \| |x|^{-1}\|^{-1}_{X^{-p}} \| |y|^{-1}\|^{-1}_{X^{-q}} = \|x\|_{X^p}\|y\|_{X^q} .$$

Finally, if $p < 0$, $q < 0$ and $r < 0$, then it follows from (3.1) that

$$\|xy\|_{X^r} = \| |xy|^{-1}\|^{-1}_{X^{-r}} \ge \| |x|^{-1}\|^{-1}_{X^{-p}} \| |y|^{-1}\|^{-1}_{X^{-q}} = \|x\|_{X^p}\|y\|_{X^q} .$$

Example 3.1. Note that it follows from b) that

$$\|xy\|_X \ge \|x\|_{X^p}\|y\|_{X^q}$$

if $0 < p < 1$ and $q = p/(p-1)$. For $X = L^1$ this is the usual reversed version of Hölder's inequality. See e.g. [10,p.146].

Example 3.2. Let $0 < p \le r \le q < \infty$ and $\frac{1}{r} = \frac{1-\theta}{p} + \frac{\theta}{q}$, $0 \le \theta \le 1$. By using a) we get

$$\|x\|_{X^r} \le \|x\|^{1-\theta}_{X^p}\|x\|^{\theta}_{X^q} .$$

In particular, this estimate implies the following additional information about the X^p-spaces: Let x be a fixed function and $f : p \to \log \|x\|_{X^p}$. Then

(1) $D_f = \{p : \|x\|_{X^p} < \infty\}$ is an interval, (or $D_f = \phi$ or $D_f = \{p_o\}$) ,

(2) $f(\frac{1}{p})$ is a convex function on D_f ,

(3) $X^p \cap X^q \subset X^r$.

If $z = 1/xy$, then

$$\|xy\|_{X^r} = \|(xy)^{-1}\|^{-1}_{X^{-r}} = \|z\|^{-1}_{X^{-r}} .$$

Hence the following generalization of an estimate obtained in [1] follows from Theorem 3.1:

Corollary 3.2. Let p, q and r be nonzero real numbers such that $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 0$ and $xyz = 1$. Then

$$\|x\|_{X^p}\|y\|_{X^q}\|z\|_{X^r} \ge 1 \tag{3.3}$$

if exactly one of p , q or r is negative. (3.3) holds in the opposite direction if exactly one of p , q or r is positive.

Remark 3.1. By using induction it is easy to generalize Corollary 3.2 (and Theorem 3.1). For example if $\pi_1^N x_i = 1$, $\Sigma_1^N \frac{1}{p_i} = 0$, $p_i \ne 0$, $i = 1, 2, 3, \ldots , N$, and exactly one of p_i is negative, then

$$\pi_1^N \|x_i\|_{X^{p_i}} \ge 1 .$$

Remark 3.2. In view of Remark 2.1 and our proof of Theorem 3.1 we find that the inequalities (3.1) - (3.3) reduce to equalities if

$$c_1|x|^p = c_2|y|^q$$

for some positive constants c_1 and c_2.

4. Minkowski's inequality. We generalize the usual Minkowski's inequality in the following way:

Theorem 4.1. Let x_i, $i = 1, 2, \ldots, N$, be measurable functions on Ω.

a) If $p \geq 1$, then

$$\|\sum_1^N x_i\|_{X^p} \leq \sum_1^N \|x_i\|_{X^p} .$$

b) If $p \leq 1$, $p \neq 0$, and

$$\|\sum_1^N x_i\|_X \geq \sum_1^N \|x_i\|_X ,$$

then

$$\|\sum_1^N x_i\|_{X^p} \geq \sum_1^N \|x_i\|_{X^p} .$$

Proof Let $p > 1$. According to Theorem 3.1 a) and Remark 2.1, we have

$$\|x\|_{X^p} = \sup_z \|xz\|_X ,$$

where supremum is taken over all non-negative functions z such that $\|z\|_{X^q} \leq 1$, $q = p/(p-1)$. Therefore

$$\|\sum_1^N x_i\|_{X^p} = \sup_z \|\sum_1^N x_i z\|_X \leq \sup_z \sum_1^N \|x_i z\|_X \leq \sum_1^N \sup_z \|x_i z\|_X = \sum_1^N \|x_i\|_{X^p} .$$

The case $p = 1$ is trivial and the proof of a) is complete. By using Theorem 3.1 b) and Remark 3.2 the proof of b) can be carried out in a similar way.

Remark 4.1. Theorem 4.1 a) can also be proved by generalizing the arguments used in [11] in the case $N = 2$.

5. Clarkson's inequalities. Our generalization of one of Clarkson's inequalities (see [7,p.400]) reads as follows.

Theorem 5.1. Let $1 < p \leq 2$, $q = p/(p-1)$, and assume that, for all non-negative functions x and y on Ω,

$$\|x\|_{X^{p-1}} + \|y\|_{X^{p-1}} \leq \|x+y\|_{X^{p-1}} .$$

Then, for all $x, y \in X^p$,

$$\left\|\frac{x+y}{2}\right\|_{X^p}^q + \left\|\frac{x-y}{2}\right\|_{X^p}^q \le \left(\frac{1}{2}\|x\|_{X^p}^p + \frac{1}{2}\|y\|_{X^p}^p\right)^{q-1} . \tag{5.1}$$

Proof Since

$$\|x\|_{X^p}^q = \||x|^q\|_{X^{p-1}}$$

we have

$$\left\|\frac{x+y}{2}\right\|_{X^p}^q + \left\|\frac{x-y}{2}\right\|_{X^p}^q = \left\|\left|\frac{x+y}{2}\right|^q\right\|_{X^{p-1}} + \left\|\left|\frac{x-y}{2}\right|^q\right\|_{X^{p-1}} \le \left\|\left|\frac{x+y}{2}\right|^q + \left|\frac{x-y}{2}\right|^q\right\|_{X^{p-1}} . \tag{5.2}$$

Moreover,

$$\left\|\left(\frac{1}{2}|x|^p + \frac{1}{2}|y|^p\right)^{q-1}\right\|_{X^{p-1}} = \left\|\frac{1}{2}|x|^p + \frac{1}{2}|y|^p\right\|_X^{1/(p-1)} \le$$

$$\le \left(\frac{1}{2}\||x|^p\|_X + \frac{1}{2}\||y|^p\|_X\right)^{1/(p-1)} = \left(\frac{1}{2}\|x\|_{X^p}^p + \frac{1}{2}\|y\|_{X^p}^p\right)^{q-1} . \tag{5.3}$$

Moreover, X^{p-1} satisfies (0.1) and the proof follows by combining (5.2) - (5.3) and using the following elementary lemma.

Lemma 5.1. Let z and w be complex numbers, $1 < p \le 2$ and $q = p/(p-1)$. Then

$$\left|\frac{z+w}{2}\right|^q + \left|\frac{z-w}{2}\right|^q \le \left(\frac{1}{2}|z|^p + \frac{1}{2}|w|^p\right)^{q-1} .$$

A proof of the lemma can be found in [7,p.400]. We also need to formulate Theorem 5.1 in the following dual form.

Theorem 5.1´. Let $2 \le p < \infty$ and $q = p/(p-1)$. Let X be a subspace of $L^0(\Omega)$ such that X^{p-1} is a Banach lattice and

$$\|x\|_X + \|y\|_X \le \|x+y\|_X$$

for all non-negative functions x and y or Ω. Then, for all $x, y \in X^p$, (5.1) holds in the opposite direction.

Proof We replace "X" by "X^{q-1}" in Theorem 5.1. Then "X^p" will be replaced by "X^q". If we also replace "x" and "y" by "$x+y$" and "$x-y$" and observe that $(p-1)(q-1) = 1$, then we see that Theorems 5.1 and 5.1´ are equivalent.

Corollary 5.2. Let $1 < p < \infty$ and $q = p/(p-1)$.

a) If $p \le 2$, then, for every $s \in [p,q]$,

$$\left\|\frac{x+y}{2}\right\|_{L^s}^q + \left\|\frac{x-y}{2}\right\|_{L^s}^q \le \left(\frac{1}{2}\|x\|_{L^s}^p + \frac{1}{2}\|y\|_{L^s}^p\right)^{q-1} . \tag{5.4}$$

b) If $p \ge 2$, then, for every $s \in [q,p]$, (5.4) holds in the opposite direction.

Proof The assumptions in Theorems 5.1 and 5.1´ holds if e.g. $X = L^a$, $1 \leq a \leq 1/(p-1)$ and $X = L^a$, $1/(p-1) \leq a \leq 1$, respectively. We put $s = pa$ and see that Corollary 5.2 is a special case of Theorems 5.1 and 5.1´.

Remark 5.1. For the case $s = p$ the inequalities in Corollary 5.2 coincide with the usual Clarkson's inequalities (see[7] or [2,p.37]). In particular, for the case $p = 2$ we obtain the usual parallelogram law for L^2-spaces.

Remark 5.2. If e.g. $\Omega = [0,1]$, μ is the Lebesque measure,

$$x(t) = \begin{cases} 2 \,, & 0 < t \leq c < 1 \,, \\ 0 \,, & c \leq t \leq 1 \,, \end{cases} \quad \text{and} \quad y(t) = \begin{cases} 0 \,, & 0 < t \leq c < 1 \,, \\ 2 \,, & c \leq t \leq 1 \,, \end{cases}$$

then we find that (5.4) reads

$$2 \leq (2^{p-1}(c^{p/s} + (1-c)^{p/s}))^{q-1} \quad \Leftrightarrow \quad 1 \leq c^{p/s} + (1-c)^{p/s} \,.$$

This inequality does not hold for any $s < p$ and we conclude that the assumption "$s \in [p,q]$" in Corollary 5.2 a) is essential. The same counterexample shows that Corollary 5.2 b) does not hold in general for any $s > p$.

Remark 5.3. F. Cobos and M. Milman have in recent papers (see [8] and [12]) generalized Clarkson's original inequalities in another direction than that presented above.

6. Uniformly convex spaces.

We say that the normed space Y is a uniformly convex space if, for any real number ε , $0 < \varepsilon \leq 2$, there exists a real number $\delta = \delta(\varepsilon) > 0$, such that the implication

$$\|x\|_Y = \|y\|_Y = 1 \,, \ \|x-y\|_Y \geq \varepsilon \ \Rightarrow \ \|\tfrac{x+y}{2}\|_Y \leq 1-\delta$$

holds for every $x , y \in Y$.

Theorem 6.1. The spaces X^p , $1 < p < \infty$, are uniformly convex if the assumptions in Theorem 5.1 or Theorem 5.1´ are satisfied.

Proof Let $\varepsilon > 0$, $\|x\|_{X^p} = \|y\|_{X^p} = 1$ and

$$\|x-y\|_{X^p} \geq \varepsilon > 0 \,.$$

If $1 < p \leq 2$ we can use Theorem 5.1 and obtain that

$$\|\tfrac{x+y}{2}\|_{X^p}^q + (\tfrac{\varepsilon}{2})^q \leq 1 \,.$$

Therefore

$$\|\tfrac{x+y}{2}\|_{X^p} \leq 1 - \delta \,, \ \delta = (1 - (1 - (\tfrac{\varepsilon}{2})^q)^{1/q} \,. \tag{6.1}$$

Let $p \geq 2$. By using Theorem 5.1´ we find in the same way that (6.1) holds with $\delta = (1 - (\frac{\varepsilon}{2})^p)^{1/p}$. The proof is complete.

Example 6.1. The assumptions in Theorem 5.1 are satisfied e.g. if $X = L^a$, $1 \leq a \leq 1/(p-1)$, $1 < p \leq 2$. We conclude that Theorem 6.1 implies the well-known fact that the spaces L^s, $1 < s < \infty$, are uniformly convex. The spaces L and L^∞ are not uniformly convex (see e.g. [2,p.38]).

7. A sharp form of Minkowski's inequality. We introduce a measure $a(x,y)$ of the deviation between the vectors $x, y \in X^p$, $p \geq 1$, in the following way:

$$a(x,y) = \| (x/\|x\|_{X^p}) - (y/\|y\|_{X^p})\|_{X^p} .$$

Theorem 7.1. Let $1 < p < \infty$, $q = p/(p-1)$, $x_i \in X^p$, $y = \Sigma_1^N x_i$, $a_i = a(x_i,y)$ and, for $i = 1, 2, \ldots, N$,

$$b_i = \begin{cases} 2(1 - (a_i/2)^q)^{1/q} - 1 & \text{if } 1 < p \leq 2 , \\ 2(1 - (a_i/2)^p)^{1/p} - 1 & \text{if } p > 2 . \end{cases}$$

If the assumptions in Theorem 5.1 or Theorem 5.1 are satisfied, then

$$\| \sum_1^N x_i \|_{X^p} \leq \sum_1^N b_i \|x_i\|_{X^p} . \tag{7.1}$$

Proof According to Theorem 6.1 we know that X^p is a uniformly convex space. Moreove we have in the proof of this theorem seen that the function $\delta = \delta(\varepsilon)$ in the definiti of uniform convexity is equal to $1 - (1-(\varepsilon/2)^q)^{1/q}$ if $1 < p \leq 2$ and equal to $1 - (1-(\varepsilon/2)^p)^{1/p}$ if $p \geq 2$. Therefore $b_i = 1 - 2\delta(a_i)$, $i = 1, 2, \ldots, N$, and (7.1) follows by using our Theorem 1.1 and Theorem 3 in [7].

Remark 7.1. It is obvious that $0 \leq a(x,y) \leq 2$, $a(x,y) = 0$ if and only if $y = cx$, $c > 0$, and $a(x,y) = 2$ if and only if $y = cx$, $c < 0$. We conclude that $-1 \leq b_i \leq 1$. Therefore we can put $\theta_i = \arccos b_i$, $i = 1, 2, \ldots, N$, and compare (7.1) with the usual formula

$$\| \sum_1^N x_i \|_E = \sum_1^N \cos \theta_i \|x_i\|_E ,$$

where x_i are vectors in $E = R^3$, $\|\cdot\|_E$ is the usual norm in the Euclidian space E an θ_i is the angle between x_i and $\Sigma_1^N x_i$.

Corollary 7.2. Let $1 < p \leq 2$, $x_i \in L^s$, $p \leq s \leq q$, $q = p/(p-1)$, $a_i = a(x_i,\Sigma_1^N x_i)$ and $b_i = 2(1-(a_i/2)^q)^{1/q} - 1$, $i = 1, 2, \ldots, N$.

Then

$$\| \sum_1^N x_i \|_{L^s} \leq \sum_1^N b_i \| x_i \|_{L^s} .$$

Proof The assumptions in Theorem 5.1 are satisfied e.g. if $X^p = L^s$, $p \leq s \leq q$, $1 < p \leq 2$ (see example 6.1). Therefore Corollary 7.2 is a special case of Theorem 7.1.

Remark 7.2. For the cases $s = p$ and $s = q$ Corollary 7.2 is already well-known (see [7,p.405]).

8. A remarkable inequality.

Our final result is the following generalization and unification of some remarkable inequalities of Beckenbach [4] and Dresher [9]. See also [5].

Theorem 8.1. *Let* $p \geq 1 \geq r > 0$, $r \neq p$. *Let* x *and* y *be positive functions satisfying*

$$\| x+y\|_{X^r} \geq \|x\|_{X^r} + \|y\|_{X^r} . \tag{8.1}$$

Then

$$\left(\frac{\| x+y \|^p_{X^p}}{\| x+y \|^r_{X^r}}\right)^{1/(p-r)} \leq \left(\frac{\|x\|^p_{X^p}}{\|x\|^r_{X^r}}\right)^{1/(p-r)} + \left(\frac{\|y\|^p_{X^p}}{\|y\|^r_{X^r}}\right)^{1/(p-r)} . \tag{8.2}$$

Proof Let $p > 1$. We choose $z \in X^q$, $q = p/(p-1)$, and put

$$a = \|zx\|_X/(\|x\|_{X^r})^{r/p} \quad \text{and} \quad b = \|zy\|_X/(\|y\|_{X^r})^{r/p} . \tag{8.3}$$

We use Hölder's inequality with conjugate exponents p/r and $p/(p-r)$ and obtain that

$$(a(\|x\|_{X^r})^{r/p} + b(\|y\|_{X^r})^{r/p})^{p/(p-r)} \leq (a^{p/(p-r)} + b^{p/(p-r)})(\|x\|_{X^r} + \|y\|_{X^r})^{r/(p-r)} .$$

Therefore, by also using (8.1) and (8.3) ,

$$(\| (x+y)z\|_X)^{p/(p-r)} \leq (\|xz\|_X + \|yz\|_X)^{p/(p-r)} =$$

$$(a(\|x\|_{X^r})^{r/p} + b(\|y\|_{X^r})^{r/p})^{p/(p-r)} \leq (a^{p/(p-r)} + b^{p/(p-r)})(\|x\|_{X^r} + \|y\|_{X^r})^{r/(p-r)} \leq$$

$$\leq \left(\left(\frac{\|zx\|^p_X}{\|x\|^r_{X^r}}\right)^{1/(p-r)} + \left(\frac{\|zy\|^p_X}{\|y\|^r_{X^r}}\right)^{1/(p-r)}\right)(\|x+y\|_{X^r})^{r/(p-r)} . \tag{8.4}$$

Furthermore, by Theorem 3.1 a) and Remark 3.2,

$$\|x\|_{X^p} = \sup_z \|xz\|_X ,$$

where supremum is taken over all positive z satisfying $\|z\|_{X^q} \leq 1$. Therefore (8.2) follows by taking supremum in (8.4). For the case $p = 1$ we let

$$a = \|x\|_X/\|x\|^r_{X^r} \quad \text{and} \quad b = \|y\|_X/\|y\|^r_{X^r}$$

and note that all estimates in (8.4) hold with $z = 1$. Therefore (8.2) holds also in this case and the proof is complete.

Remark 8.1. By analysing our proof we see that Theorem 8.1 also holds without the restriction (8.1) if "$\|x+y\|_{X^r}$" is replaced by "$\|x\|_{X^r} + \|y\|_{X^r}$" .

Remark 8.2. For the case $X = \ell^{1,N}$ and $r = p-1$ (8.2) coincides with a well-known inequality of Beckenbach, see [4] and also [5,p.27].

Example 8.1. In particular, (8.1) holds if $X = L^b, 0 < b \le \frac{1}{r}$. Therefore, if $p \ge 1 \ge r > 0$, $r \ne p$, $r \le s \le 1$ and $a = ps/r$, then

$$\left(\frac{\|x+y\|^p_{L^a}}{\|x+y\|^r_{L^s}}\right)^{1/(p-r)} \le \left(\frac{\|x\|^p_{L^a}}{\|x\|^r_{L^s}}\right)^{1/(p-r)} + \left(\frac{\|y\|^p_{L^a}}{\|y\|^r_{L^s}}\right)^{1/(p-r)} .$$

For the case $s = r$ this is an inequality of Dresher [9,p.267]. However, our proof is quite different and more elementary.

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