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TWO WEIGHTS WEAK TYPE INEQUALITY FOR THE MAXIMAL FUNCTION IN $L(1 + \log^+ L)^K$

Luboš PICK

1. Introduction

For $f \in L_{1,loc}(\mathbb{R}^n)$ the *Hardy-Littlewood maximal operator* is defined by

$$Mf(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(t)| \, dt$$

where supremum is taken over all cubes with sides parallel to the coordinate axes. Let further ρ and σ be *weights*, i.e. measurable and a.e. positive and finite functions on $\mathbb{R}^n$. For $K \geq 1$, let Φ_K denote the Young function

$$\Phi_K(t) = t \cdot (1 + \log^+ t)^K , \quad t \geq 0 ,$$

generating the Orlicz space $L_{\Phi_K}(\mathbb{R}^n)$. In this note a necessary and sufficient condition in order that M maps continuously the weighted Orlicz space $L_{\Phi_K,\sigma}$ into the weak weighted Orlicz space $L^*_{\Phi_K,\rho}$ is given.

In his fundamental paper [4], B. MUCKENHOUPT showed that for $1 < p < \infty$ the weak type inequality

$$\rho(\{Mf > \lambda\}) \leq C \cdot \int \left(\frac{|f(x)|}{\lambda}\right)^p \sigma(x) \, dx , \quad f \in L_p , \tag{1.1}$$

holds iff the couple (ρ,σ) satisfies the well-known (A_p) condition

$$\sup_Q \left(\frac{1}{|Q|} \int_Q \rho(x) \, dx\right)^{1/p} \cdot \left(\frac{1}{|Q|} \int_Q \sigma(x)^{-1/(p-1)} \, dx\right)^{1/p'} = A_p(\rho,\sigma) < \infty$$

which can be rewritten as

$$\sup_Q \left(\int_Q \left(\frac{\rho(Q)}{|Q|\sigma(x)}\right)^{p'} \frac{\sigma(x)}{\rho(Q)} \, dx\right)^{1/p'} = A_p(\rho;\sigma) < \infty \tag{1.2}$$

where $p' = p/(p-1)$. In the same paper there was also obtained the relation between the best right hand side constants in (1.1) and (1.2)

(1.3) $$[A_p(\rho,\sigma)]^p \leq C \ .$$

The limiting case of (1.2) is the condition

(1.4) there exists $\mu > 0$ such that
$$\sup_Q \int_Q \exp\left(\frac{\mu\rho(Q)}{\sigma(x)|Q|}\right) \frac{\sigma(x)}{\rho(Q)}\, dx = A_{\log}(\rho,\sigma) < \infty$$

which was investigated by M. KRBEC in [3]. He proved the necessity and sufficiency of (1.4) for M to be continuous from the weighted Zygmund class $L(1+\log^+ L)_\sigma$ to the weak weighted Zygmund class $L(1+\log^+ L)^*_\rho$. The goal of this note is to generalize KRBEC's result to more general nonreflexive Orlicz spaces, namely to the spaces L_{Φ_K} .

2. Preliminaries

We shall use the following definition and lemmas.

DEFINITION 2.1. *The Young function* M *satisfies the* Δ^2 *condition if there exist positive constants* u_0 , B *such that for all* $u \geq u_0$

$$M^2(u) \leq M(Bu) \ .$$

LEMMA 2.2. *Let function* F_K *be defined for* $K \geq 1$ *by*

(2.1) $$F_K(z) = \sum_{j=0}^{\infty} \frac{z^j}{j!} \left[\frac{1}{j-1}\right]_*^{j(K-1)} \ , \quad z \geq 1 \ ,$$

(*where* $[\frac{a}{b}]_* = 1$, *if* $ab \leq 0$, *otherwise* $[\frac{a}{b}] = \frac{a}{b}$),

$$F_K(0) = 0 \quad \textit{and} \quad F_K \ \textit{linear on} \ (0,1) \ .$$

Then F_K *satisfies the* Δ^2 *condition.*

P r o o f . We need to show the existence of $B > 0$ such that the inequality

$$\sum_{\ell=0}^{j} \binom{j}{\ell} \left\{\frac{[j-1]_*^j}{[(j-\ell)-1]_*^{j-\ell}[\ell-1]_*^{\ell}}\right\}^{K-1} \leq B^j$$

holds for each $j \geq 0$. Denoting the term in the curly brackets by $R_{j\ell}$ we have for $j \geq 4$ the estimate

(2.2) $$R_{j\ell} \leq 4^j \ .$$

Now it is easy to verify (2.2) for the remaining values of j . Setting $B = 4^K$ we finish the proof.

LEMMA 2.3. *Let* M , N *be two complementary Young functions,* M *satisfying the* Δ^2 *condition. Then there exist* $\beta > 0$ *and* $v_0 \geq 0$ *such that for every* $v \geq v_0$

$$N(v) \leq \beta v \cdot M^{-1}(\beta v) \ . \tag{2.3}$$

(For the proof see [2], Theorem 6.1.)

3. The result

THEOREM 3.1. *Let* ρ *and* σ *satisfy*

$$\sup_Q \frac{\sigma(Q)}{\rho(Q)} = H < \infty \ . \tag{3.1}$$

Then the following statements are equivalent.

(3.2) i) $$\rho(\{Mf > \lambda\}) \leq C_{(\rho,\sigma,K,n)} \int \Phi_K\left(\frac{|f(x)|}{\lambda}\right) \sigma(x)\, dx \ , \quad f \in L_{\Phi_K} \ , \quad \lambda > 0 \ .$$

ii) *There exists* $\mu > 0$ *such that*

$$\sup_Q \sum_{j=0}^{\infty} \frac{\mu^j}{j!} \left[\frac{1}{j-1}\right]_*^{j(K-1)} \int_Q \left(\frac{\sigma(Q)}{|Q|\sigma(x)}\right)^j \frac{\sigma(x)}{\rho(Q)}\, dx = A^K_{\log} < \infty \tag{3.3}$$

P r o o f . The proof of (3.1) $\Longrightarrow$ (3.2) goes in a similar way as in [3] so that just a sketch will be given.

First of all, it is easy to check that for all $z \in (\frac{1}{2},\infty)$ and $p \in (1,2)$ we have

$$\Phi_K(z) \leq C_K \frac{z^p}{(p-1)^K} \ . \tag{3.4}$$

Using the standard argument, (3.2) and (3.4) yield

$$\rho(\{Mf > \lambda\}) \leq C_{(\rho,\sigma,K,n)} \int_{\{|f| > \lambda/2\}} \Phi_K\left(\frac{|f(x)|}{\lambda}\right) \sigma(x)\, dx$$

$$\leq C_K\, C_{(\rho,\sigma,K,n)} \frac{1}{(p-1)^K} \int_{\{|f| > \lambda/2\}} \left(\frac{|f(x)|}{\lambda}\right)^p \sigma(x)\, dx \ .$$

Now, by MUCKENHOUPT's relation (1.3), $A_p(\rho,\sigma)$ satisfies (see (1.2))

$$A_p(\rho,\sigma) \leq \tilde{C}_{(K,\rho,\sigma,n)} \frac{p'}{(p-1)^{K-1}} \ .$$

So we can write

$$\sum_{j=0}^{\infty} \frac{\mu^j}{j!} \left[\frac{1}{j-1}\right]_*^{j(K-1)} \int_Q \left(\frac{\rho(Q)}{|Q|\sigma(x)}\right)^j \frac{\sigma(x)}{\rho(Q)}\, dx$$

$$\leq \frac{\sigma(Q)}{\rho(Q)} + \mu + \sum_{j=2}^{\infty} \frac{\mu^j}{j!} \left[\frac{1}{j-1}\right]^{j(K-1)} [A_{j'}(\rho,\sigma)]^j \leq H + \mu + \sum_{j=2}^{\infty} \frac{\mu^j}{j!} \tilde{C}^j_{(K,\rho,\sigma,n)}\, j^j$$

which is finite for $0 < \mu < (\tilde{C}\cdot e)^{-1}$.

(3.2) implies (3.1).

Comparing the right hand side series in (2.1) with the Taylor expansion of exponential function we get after rather elementary but tedious calculation that for $u \geq 1$

$$F_K(u) > \exp\left(\frac{1}{2} u^{1/K}\right) .$$

Now let G_K denote the function complementary to F_K . Lemma 2.3 provides the existence of some $\alpha_K > 0$ and $v_0 \geq 0$ such that

$$G_K(v) \leq \alpha_K\cdot 2^K(\log \alpha_K + \log v)^K\cdot v , \quad v \geq v_0 ,$$

which can be rewritten as

$$G_K(v) \leq C_K\cdot\Phi_K(v) , \quad v \geq v_0 .$$

The Young inequality and the last inequality give

$$\begin{aligned}\frac{1}{|Q|}\int_Q |f(t)|\, dt &\leq \tilde{C}_K\cdot\int_Q F_K\left(\frac{\mu\cdot\rho(Q)}{|Q|\cdot\sigma(x)}\right)\frac{\sigma(x)}{\rho(Q)}\, dx \\ &\quad + C_K\cdot C_\mu\cdot\int_Q \Phi_K(|f(x)|)\frac{\sigma(x)}{\rho(Q)}\, dx \\ &\leq \tilde{C}_K\cdot A^K_{\log} + C_{K\mu}\cdot I_{Q,K}(f) .\end{aligned} \tag{3.5}$$

The well known Besicovitch covering argument guarantees the existence of a sequence of cubes $\{Q_j\}$ satisfying

$$\begin{aligned}&\text{(i)} \quad I_{Q_j,K}(f) > 1 , \\ &\text{(ii)} \quad \bigcup_{j=1}^{\infty} Q_j \supset \{\sup_Q I_{Q,K}(f) > 1\} , \\ &\text{(iii)} \quad \Big\|\sum_{j=1}^{\infty}\chi_{Q_j}(x)\Big\|_\infty \leq C_n .\end{aligned} \tag{3.6}$$

Combining all statements in (3.6) we obtain

$$\rho\left(\{Mf > C_{K\mu} + \tilde{C}_K\cdot A^K_{\log}\}\right) \leq C_n\cdot\int\Phi_K(|f(x)|)\sigma(x)\, dx \tag{3.7}$$

Now, for $\lambda > 0$ arbitrary, (3.7) implies

$$\begin{aligned}\rho(\{Mf > \lambda\}) &\leq C_n\cdot\int\Phi_K\left(\frac{\tilde{C}_K\cdot A^K_{\log} + C_{K\mu}}{\lambda}|f(x)|\right)\sigma(x)\, dx \\ &\leq \tilde{C}_{n,K,\mu}\cdot\int\Phi_K\left(\frac{|f(x)|}{\lambda}\right)\sigma(x)\, dx .\end{aligned}$$

References

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Mathematical Institute
Czechoslovak Academy of Sciences
Žitná 25
115 67 Praha 1
Czechoslovakia