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TRIGONOMETRIC APPROXIMATION BY EULER MEANS

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1. Introduction. Let X be one of the spaces L^p $(1\le p<\infty)$ or C of 2π-periodic complex-valued functions with norm

$$\|f\|_p = \begin{cases} \left(\frac{1}{2\pi}\int_0^{2\pi}|f(x)|^p\,dx\right)^{1/p} & \text{if } 1\le p<\infty, \\ \max_x |f(x)| & \text{if } p=\infty \quad (X=C). \end{cases}$$

By $X^{r,\beta}$ $(r=0,1,2,\ldots,\ 0\le\beta\le 1)$ we denote the Lipschitzspace of all functions f, for which there exists a (r-1)-times absolutely continuous function g with $g^{(r)}\in X$, f=g in X and

$$(1)\qquad \sup_{|h|\le\delta}\|g^{(r)}(\cdot+h)-g^{(r)}(\cdot)\|_p = O(\delta^\beta) \qquad (\delta\to 0+).$$

Let be $f\in\tilde{X}^{r,\beta}\subset X^{r,\beta}$ $(0\le\beta<1)$, if (1) is fulfilled with "small-o". Furthermore, by A_M^a we denote the set of functions $f\in C$, which have an analytic extension onto the strip $S=\{z:|\mathrm{Im}\,z|<a\}$ with $|f(z)|\le M$ if $z\in S$ (see also [3]). The n-th partial sum of the Fourier series of $f\in X$ is given by

$$S_nf(x) = \sum_{k=-n}^{n} f^\wedge(k)\,e^{ikx} .$$

Then we define the Euler means of $f\in X$ for $0<q<\infty$ by

$$T_{n,q}f = (1+q)^{-n}\sum_{k=0}^{n}\binom{n}{k}q^{n-k}S_kf .$$

The best approximation of f in X is denoted by

$$E_n(f,X) = \inf_{p_n\in T_n}\|f-p_n\|_p,$$

where T_n is the set of trigonometric polynomials of degree $\leq n$. For $x_k = 2k\pi/(2n+1)$ let $L_n f \in T_n$ be uniquely determined by $L_n f(x_k) = f(x_k)$, $k=0,\dots,2n$. Then we define the Euler means of the interpolatory polynomial by $L_{n,q} f = T_{n,q} L_n f$.

Euler summability of Fourier series has been discussed by several authors. Chui, Holland et al. [1], [6] obtained convergence theorems provided that a certain integrability condition is imposed. These results were generalized by Ting-fan [10], Efimov [2] and Pych-Taberska [9]. It is the purpose of this note to show that the Fourier sum and their Euler means yields the same order of convergence for various function classes. Thus we can improve results from [4] and [5]. However, this is not true for analytic functions, since the Euler means are saturated with exponential order. The method of proof gives a unified approach to the L^p- and C-norm and allows to study Euler means of interpolatory processes.

2. Convergence for $f \in X^{r,\beta}$.

To prove Theorem 1 we need the following estimate.

Lemma 1. For $d \geq 2$ and $n \in \mathbb{N}$,

$$\binom{n}{[\frac{n}{d}]} \leq \left(\frac{d}{n}\right)^{1/2} \left(\frac{d}{d-1}(d-1)^{1/d}\right)^n .$$

Proof. For $n<d$ the inequality is trivial. Otherwise we set $[\frac{n}{d}] = \frac{n}{z}$. This yields $d \leq z \leq 2d$. Stirling's formula gives now

$$\binom{n}{\frac{n}{z}} = \frac{n!}{(\frac{n}{z})!\,(n-\frac{n}{z})!} \leq e^{1/12} \cdot \frac{n^n}{(\frac{n}{z})^{n/z}(n-\frac{n}{z})^{n-n/z}} \left(\frac{z}{2\pi(n-\frac{n}{z})}\right)^{1/2}$$

$$= \frac{e^{1/12}\cdot z}{(2\pi n(z-1))^{1/2}} \left(z(z-1)^{(1-z)/z}\right)^n$$

$$\leq 0{,}5 \cdot \frac{2d}{(n(2d-1))^{1/2}} \cdot \left(\frac{d}{d-1}(d-1)^{1/d}\right)^n ,$$

from which the assertion follows immediately. ∎

Theorem 1. For $f \in X^{r,\beta}$ and $n \to \infty$,

$$\|f - T_{n,q} f\|_p = \begin{cases} O(n^{-r-\beta}\log n) & \text{if } p=1 \text{ or } p=\infty, \\ O(n^{-r-\beta}) & \text{if } 1<p<\infty . \end{cases}$$

"Large-O" can be replaced by "small-o", if $f\in\tilde{X}^{r,\beta}$, $0\le\beta<1$.

Proof. Let

$$(2)\qquad t=\frac{d}{d-1}\cdot\frac{(d-1)^{1/d}}{q+1}\cdot\begin{cases}q & \text{if } q\ge 1\ ,\\ 1 & \text{if } 0<q<1\ .\end{cases}$$

It is easy to see that we can choose $d=d(q)>2$ such that $t<1$ holds. With $m=[n/d]$ we estimate

$$\|f-T_{n,q}f\|_p\le(q+1)^{-n}\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\|f-S_kf\|_p$$

$$\le(q+1)^{-n}\Big(\sum_{k=0}^{m-1}\binom{n}{k}q^{n-k}\max_{0\le j\le m-1}\|f-S_jf\|_p+$$

$$+\sum_{k=m}^{n}\binom{n}{k}q^{n-k}\max_{m\le j\le n}\|f-S_jf\|_p\Big)\ .$$

Using the well-known inequalities

$$\|f-S_jf\|_p\le(1+\|S_j\|_{X\to X})E_j(f,X)$$

and $\|S_j\|_{X\to X}\le C(p,j)$ with

$$(3)\qquad C(p,j)=\begin{cases}1,5+4\pi^{-2}\ln j & \text{if } p=1 \text{ or } p=\infty\ ,\\ c_p & \text{if } 1<p<\infty\ ,\end{cases}$$

we get by the monotonicity of E_j

$$\|f-T_{n,q}f\|_p\le 2C(p,m)\cdot\|f\|_p(q+1)^{-n}\sum_{k=0}^{m-1}\binom{n}{k}q^{n-k}+$$

$$+2C(p,n)E_m(f,X)=S_1+S_2\ .$$

Jackson's theorem and (3) imply the assertion for the second term S_2. Remark that d does not depend on n which yields

$$(n/m)^{r-\beta}=O(1)\qquad(n\to\infty)\ .$$

To estimate the term S_1 we use Lemma 1. If $0<q<1$ we obtain

$$\sum_{k=0}^{m-1}\binom{n}{k}q^{n-k}<m\binom{n}{m}<\left(\frac{n}{d}\right)^{1/2}\left(\frac{d}{d-1}(d-1)^{1/d}\right)^{n}\ ,$$

and otherwise for $q\ge 1$

$$\sum_{k=0}^{m-1} \binom{n}{k} q^{n-k} < m \binom{n}{m} q^n < \left(\frac{n}{d}\right)^{1/2} \left(\frac{dq}{d-1}(d-1)^{1/d}\right)^n .$$

In both cases it follows from (2)

$$S_1 < 2C(p,m) \|f\|_p \, m^{1/2} t^n .$$

This means that S_1 tends to zero with exponential order and therefore the theorem is proved. ■

3. Analytic functions.

Theorem 2. Let $f \in A_M^a$. Then

$$\|f - T_{n,q} f\|_p \le \frac{2M}{e^a - 1} \left(\frac{q + e^{-a}}{q+1}\right)^n .$$

Proof. For $0 < |j| \le n$ we compute easily

$$(4) \qquad (f - T_{n,q} f)^\wedge(j) = (q+1)^{-n} f^\wedge(j) \sum_{k=0}^{|j|-1} \binom{n}{k} q^{n-k}$$

and $(f - T_{n,q} f)^\wedge(0) = 0$. Hence,

$$\|f - T_{n,q} f\|_p \le \sum_{|j|>n} |f^\wedge(j)| + \sum_{\substack{j=-n \\ j \ne 0}}^{n} (q+1)^{-n} |f^\wedge(j)| \sum_{k=0}^{|j|-1} \binom{n}{k} q^{n-k} .$$

Using

$$(5) \qquad |f^\wedge(j)| \le M e^{-a|j|} \qquad \text{for } f \in A_M^a,\ j \in Z,$$

we get

$$\|f - T_{n,q} f\|_p \le 2M \frac{e^{-a(n+1)}}{1 - e^{-a}} + \sum_{j=1}^{n} \frac{2Me^{-aj}}{(q+1)^n} \sum_{k=0}^{j-1} \binom{n}{k} q^{n-k} .$$

Theorem 2 follows now by Abel transform and simple calculations. ■

In [6] it was stated as a problem to find the saturation class of the Euler means. The answer is given in the following theorem.

Theorem 3. The saturation class of the Euler means is T_1 and the saturation order is $(q/(q+1))^n$. That means $f \in T_1$ iff

$$(6) \qquad \|f - T_{n,q} f\|_p = O\left(\left(\frac{q}{q+1}\right)^n\right) \qquad (n \to \infty) .$$

Furthermore,

(7) $\|f - T_{n,q}f\|_p = o((\frac{q}{q+1})^n)$ $(n\to\infty)$

implies $f = \text{const}$.

Proof. A direct computation shows that it holds for $f \in T_1$

$$\|f - T_{n,q}f\|_p = (\frac{q}{q+1})^n \|f - S_0 f\|_p ,$$

which gives (6). Now let (6) be fulfilled and assume $f \notin T_1$. Then there exists a j with $f^\wedge(j) \neq 0$ and $|j| > 1$. Choosing $n \geq |j|$ we obtain by

(8) $|(f - T_{n,q}f)^\wedge(j)| \leq \|f - T_{n,q}f\|_p$

and (4) that

$$\sum_{k=0}^{|j|-1} \binom{n}{k} q^{-k} = O(1) \qquad (n\to\infty) ,$$

which contradicts the assumption $|j| > 1$. Supposing now (7), we get by (4) and (8) for $j \neq 0$ the condition

$$|f^\wedge(j)| \sum_{k=0}^{|j|-1} \binom{n}{k} q^{-k} = o(1) \qquad (n\to\infty) .$$

This yields immediately $f^\wedge(j) = 0$ for $j \neq 0$. ■

4. Interpolation. For the interpolatory polynomial $L_{n,q}f$ we obtain the following convergence result.

Theorem 4. Let $f \in C^{r,\beta}$. Then for $n\to\infty$

$$\|f - L_{n,q}f\|_p = \begin{cases} O(n^{-r-\beta}) & \text{if } 1 \leq p < \infty , \\ O(n^{-r-\beta}\log n) & \text{if } p = \infty . \end{cases}$$

If $f \in \tilde{C}^{r,\beta}$, then "large-O" can be replaced by "small-o".

The proof follows the line of Theorem 1 using

$$\|L_n\|_{C\to X} \leq \begin{cases} 2 + \frac{2}{\pi}\ln n & \text{if } p = \infty , \\ 3c_p & \text{if } 2 < p < \infty , \\ 1 & \text{if } 1 \leq p \leq 2 . \end{cases}$$

Another method of proof is to apply the following lemma, which is interesting for itself, too.

Lemma 2. For $f \in C$ we have

$$(9) \qquad \|L_{n,q}f - T_{n,q}f\|_p \le (\|L_{n,q}\|_{C\to X} + \|T_{n,q}\|_{C\to X})\, E_n(f,C)$$

with

$$\|T_{n,q}\|_{C\to X} \le \begin{cases} 1,5 + 4\pi^{-2}\ln n & \text{if } p = \infty, \\ c_p & \text{if } 2<p<\infty, \\ 1 & \text{if } 1\le p\le 2 \end{cases}$$

and

$$\|L_{n,q}\|_{C\to X} \le \begin{cases} 6,3 + 1,7\ln n & \text{if } p = \infty, \\ 3c_p^2 & \text{if } 2<p<\infty, \\ 1 & \text{if } 1\le p\le 2 . \end{cases}$$

Proof. Let $E_n(f,C) = \|f - p_n\|_\infty$. Then we obtain (9) from $L_{n,q}p_n = T_{n,q}p_n$, which implies

$$\|L_{n,q}f - T_{n,q}f\|_p \le \|L_{n,q}(f-p_n)\|_p + \|T_{n,q}(p_n-f)\|_p .$$

The estimates of the operatornorms we get by the following elementary calculations

$$\|T_{n,q}f\|_p \le (q+1)^{-n}\sum_{k=0}^{n}\binom{n}{k} q^{n-k}\,\|S_k\|_{X\to X}\|f\|_p$$

$$\le \max_{0\le k\le n}\|S_k\|_{X\to X}\|f\|_p ,$$

$$\|T_{n,q}f\|_p \le \|T_{n,q}f\|_2 \qquad \text{if } 1\le p\le 2,$$

$$\|L_{n,q}f\|_p \le \|T_{n,q}\|_{X\to X}\|L_nf\|_p$$

and

$$\|L_{n,q}\|_{C\to C} \le (1+\pi)\|T_{n,q}\|_{C\to C} . \quad \blacksquare$$

Let us remark here that it holds for $f\in X^{r,0}$, $r\ge 1$

$$\|L_{n,q}f\|_p \le c(r,p)\cdot(\|f\|_p + \|f^{(r)}\|_p)\cdot\begin{cases} \ln n & \text{if } p=1 \text{ or } p=\infty , \\ 1 & \text{if } 1<p<\infty . \end{cases}$$

The **proof** is similar to the corresponding considerations in [7] and [8, chap. 4]. We omit the details. Using

$$\|f - L_{n,q}f\|_p \le \|f - T_{n,q}f\|_p + \|T_{n,q}(f-t_n)\|_p + \|L_{n,q}(t_n-f)\|_p ,$$

where $E_n(f,X) = \|f - t_n\|_p$, we get for the interpolation error of $f \in X^{r,\beta}$ or $f \in \tilde{X}^{r,\beta}$, $r \ge 1$, the same estimates as in Theorem 1. Let now $f \in A_M^a$. For the discrete Fourier coefficients

$$f_n^\wedge(j) = \frac{1}{2n+1} \sum_{k=0}^{2n} f(x_k) \exp(-ijx_k)$$

we have by the formula of Euler (also called "method of aliasing")

$$f_n^\wedge(j) = \sum_{r=-\infty}^{\infty} f^\wedge(j + r(2n+1)) .$$

Thus, from (5),

$$|f_n^\wedge(j)| \le M\Big(e^{-a|j|} + \sum_{r=1}^{\infty} e^{-a(r(2n+1)+j)} + e^{-a(r(2n+1)-j)} \Big)$$

$$= M e^{-a|j|} \cdot \frac{2e^{a(2n+1)}}{e^{a(2n+1)}-1} .$$

For sufficiently large n we get

$$|f_n^\wedge(j)| \le 3Me^{-a|j|} .$$

Now we write for $f \in A_M^a$

$$f(x) = \sum_{k=-\infty}^{\infty} f_n^\wedge(k)\, e^{ikx} .$$

Then the following result has the same proof as the corresponding Theorems 2 and 3.

<u>Theorem 5</u>. For $f \in A_M^a$, $1 \le p \le \infty$ and sufficiently large n,

$$\|f - L_{n,q}f\|_p \le \frac{6M}{e^a-1} \left(\frac{q+e^{-a}}{q+1}\right)^n .$$

Furthermore, the set of operators $L_{n,q}: C \to X$ is saturated with order $(q/(q+1))^n$ and the saturation class is T_1.

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