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BIVARIATE HERMITE SPLINE INTERPOLATION

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1. Introduction. In this paper the method of Boolean interpolation, developed by Gordon [3], is used to solve the *reduced Hermite spline interpolation problem* of bivariate 1-periodic functions on equidistant meshes. An error estimate in the uniform norm is given in chapter 4.

2. Some Results in Univariate Hermite Spline Interpolation.

As the construction of our bivariate interpolator is based upon the univariate interpolation we start with some properties of this case. Let $C_1^u(\mathbb{R})$ denote the set of all 1-periodic u-times continuously differentiable real functions, and let $\{ x_j = j/N : j \in Z \}$ define a uniform mesh.

Definition: For integers q, r, $0 \le q \le r-1 < N(q+1)$, a function $s \in C_1^{2r-q-2}(\mathbb{R})$ with $D^{2r}s(x) = 0$ ($x \ne x_j$, $j \in Z$), is called a 1-periodic spline of order $2r-1$ and deficiency $q+1$. We denote the space of these functions S^q.

It is well known that for every $f \in C_1^q(\mathbb{R})$ there exists a unique $s =: P^q f \in S^q$ that solves the periodic Hermite spline interpolation problem $D^u P^q f(x_j) = D^u f(x_j)$, $0 \le u \le q$ and $j \in Z$. For the computation of $P^q f$ see [4].

Let us consider the projectors $P^q : C_1^q(\mathbb{R}) \to S^q$, $q = 0,\dots,r-1$. Since $S^q \subseteq C_1^{r-1}(\mathbb{R})$ we are able to compose the Hermite spline projectors of equal order and different deficiency.

Proposition 1: *For integers* q,t *with* $0 \le q,t \le k < r$ *the projectors* P^q *and* P^t *commute on* $C_1^k(\mathbb{R})$, *i. e.*

$$P^q P^t = P^t P^q = P^{\min(q,t)} .$$

Proof: Without loss of generality we assume $q \le t$. Then we obtain $C_1^{2r-q-2}(\mathbb{R}) \subseteq C_1^{2r-t-2}(\mathbb{R})$ and $P^q f \in S^q \subseteq S^t$, and the function $P^q f$ is fix under the projector P^t which proves the equation on the right. On the other hand with $D^u P^q(P^t f)(x_j) = D^u P^t f(x_j) = D^u f(x_j)$ for $0 \le u \le q \le t$, $j \in Z$, the interpolation conditions of $P^q f$ are fulfilled by $P^q P^t f \in S^q$, and with unicity of the interpolating spline in S^q the proof is completed.

The following error estimation in the uniform norm can be proved by r-fold use of the Rayleigh-Ritz inequality on $\|f - P^q f\|_2$, combined with the second integral relation of Hermite spline interpolation and with the Cauchy-Schwarz inequality (c.f.[2]).

Proposition 2: *Given* $f \in C_1^{w+1}(\mathbb{R})$ *with* $w + 1 \le 2r$. *Then*

$$\|f - P^q f\|_\infty \le K(q,w)\ h^{w+1/2}\ \|D^{w+1} f\|_\infty ,$$

where $K(q,w)$ *is a constant independent of* $h := N^{-1}$ *and* f.

2. Boolean Interpolation. Let $C_{1,1}^k(\mathbb{R}^2)$ be the set of all bivariate 1-periodic functions $F(x,y)$ with continuous mixed derivates $D_1^u D_2^v F$ for $0 \le u+v \le k$. Our purpose is to solve the reduced periodic Hermite interpolation problem, that is to construct a bivariate spline function which interpolates F and its above mentioned derivatives in every knot $(x_i, y_j) = (i/N, j/N)$, $i,j \in Z$. In the sequel we fix k,r,N with $1 \le k < r \le N$.

Let us consider for $0 \le t,w \le k$ the unique parametric extensions P_x^t with y as a parameter and P_y^w with parameter x of the univariate projectors to the bi-1-periodic space $C_{1,1}^k(\mathbb{R}^2)$. It is well known that they fulfill the interpolation properties

$$D_1^u P_x^t\ F(x_i,y) = D_1^u F(x_i,y) \quad \text{for } i \in Z,\ y \in \mathbb{R} \text{ and } 0 \le u \le t,$$

and

$$D_2^v P_y^w F(x,y_j) = D_2^v F(x,y_j) \quad \text{for } j \in Z\ \ x \in \mathbb{R} \text{ and } 0 \le v \le w.$$

Thus with $u \le t$ and $v \le w$ the tensor product yields the interpolation property

$$D_1^u D_2^v [P_x^t P_y^{k-t} F](x_i,y_j) = D_1^u D_2^v F(x_i,y_j) \quad \text{for all } i,j \in Z.$$

Since the parametric extensions mutually commute, the above tensor products also are commuting projectors and we can define the Boolean sums (cf. [3])

$$P_x^t P_y^{k-t} \oplus P_x^q P_y^{k-q} := P_x^t P_y^{k-t} + P_x^q P_y^{k-q} - [P_x^t P_y^{k-t}][P_x^q P_y^{k-q}]$$
$$= P_x^t P_y^{k-t} + P_x^q P_y^{k-q} - P_x^{\min(t,q)} P_y^{k-\max(t,q)},$$

for $0 \le t,q \le k$, and the generalisation to more than two summands, especially

$$\bigoplus_{t=0}^{k} P_x^t P_y^{k-t} := P_x^0 P_y^k \oplus (P_x^1 P_y^{k-1} \oplus (\ldots \oplus P_x^k P_y^0)\ldots)$$
$$= \sum_{t=0}^{k} P_x^t P_y^{k-t} - \sum_{t=0}^{k-1} P_x^t P_y^{k-t-1} \quad \text{(cf. [1])}.$$

It is well known that the Boolean sum gathers all the interpolation properties of its summands:

$$D_1^u D_2^v \Big[\bigoplus_{t=0}^{k} P_x^t P_y^{k-t}\Big] F(x_i,y_j) = D_1^u D_2^v \; P_x^u P_y^{k-u} \Big[\bigoplus_{t=0}^{k} P_x^t P_y^{k-t}\Big] F(x_i,y_j)$$
$$= D_1^u D_2^v \; P_x^u P_y^{k-u} F(x_i,y_j) = D_1^u D_2^v F(x_i,y_j)$$

for $0 \le u \le k$ and $0 \le v \le k-u$.

__Theorem 1:__ *The projection* $\bigoplus_{t=0}^{k} P_x^t P_y^{k-t} F$ *solves the reduced periodic Hermite interpolation problem*

$$D_1^u D_2^v \Big[\bigoplus_{t=0}^{k} P_x^t P_y^{k-t}\Big] F(x_i,y_j) = D_1^u D_2^v F(x_i,y_j)$$

for $0 \le u+v \le k$ *and* $i,j \in \mathbb{Z}$.

3. Error estimates for the bivariate case.

The remainder projector

$$I - \bigoplus_{t=0}^{k} P_x^t P_y^{k-t} = \sum_{t=0}^{k} (I - P_x^t P_y^{k-t}) - \sum_{t=0}^{k-1} (I - P_x^t P_y^{k-t-1}),$$

where I is the identity operator, is a sum of tensor product remainders, and thus equals

$$\sum_{t=0}^{k} \{(I-P_x^t) + (I-P_y^{k-t}) - (I-P_x^t)(I-P_y^{k-t})\}$$
$$-\sum_{t=0}^{k-1} \{(I-P_x^t) + (I-P_y^{k-t-1}) - (I-P_x^t)(I-P_y^{k-t-1})\},$$

$$= (I-P_x^k) + (I-P_y^k) - \sum_{t=0}^{k} (I-P_x^t)(I-P_y^{k-t}) + \sum_{t=0}^{k-1} (I-P_x^t)(I-P_y^{k-t-1}),$$

using the Boolean structure of those (cf. [1]).

We easily estimate the remainders of the parametric extensions for functions $F \in C^{2r}_{1,1}(\mathbb{R}^2)$ using proposition 2 with $w = 2r-1$ and $q = k$:

$$\|F - P^k_x F\|_\infty := \max_{x,y\in\mathbb{R}} |F(x,y) - P^k_x F(x,y)| = \max_{y\in\mathbb{R}} (\max_{x\in\mathbb{R}} |F(x,y) - P^k_x F(x,y)|)$$

$$\le \max_{y\in\mathbb{R}} (h^{2r-1/2} K(k,2r-1) \max_{x\in\mathbb{R}} |D^{2r}_1 F(x,y)|)$$

$$= h^{2r-1/2} K(k,2r-1) \|D^{2r}_1 F\|_\infty \le h^{2r-1} K(k,2r-1) \|D^{2r}_1 F\|_\infty ,$$

and also

$$\|F - P^k_y F\|_\infty \le h^{2r-1/2} K(k,2r-1) \|D^{2r}_2 F\|_\infty \le h^{2r-1} K(k,2r-1) \|D^{2r}_2 F\|_\infty .$$

For $F \in C^{2r}_{1,1}(\mathbb{R}^2) \subseteq C^{r,r}_{1,1}(\mathbb{R}^2)$ the error term products yield

$$\|(I-P^t_x)(I-P^{q-t}_y)F\|_\infty = \max_{x,y\in\mathbb{R}} |(I-P^t_x)(I-P^{q-t}_y)F(x,y)|$$

$$\le \max_{y\in\mathbb{R}} h^{r-1/2} K(t,r-1) \max_{x\in\mathbb{R}} |D^r(I-P^{q-t}_y)F(x,y)|$$

$$= h^{r-1/2} K(t,r-1) \max_{x,y\in\mathbb{R}} |(I-P^{q-t}_y)D^r_x F(x,y)|$$

$$\le h^{r-1/2} K(t,r-1)\, h^{r-1/2} K(q-t,r-1) \max_{x,y\in\mathbb{R}} |D^r_y D^r_x F(x,y)|$$

$$= h^{2r-1} K(t,r-1) K(q-t,r-1) \|D^r_x D^r_y F\|_\infty \quad ,\text{with } q \in \{k-1, k\} ,$$

since $D^r_x F$, considered as a function in y, lies in $C^r_1(\mathbb{R})$.

With $K := \max [\{ K(t,r-1) K(q-t,r-1) : 0\le t\le q,\ q \in \{k-1, k\} \} \cup \{ K(k,2r-1) \}]$ the error estimate of the Boolean sum yields now

$$\| F - \bigoplus_{t=0}^{k} P^t_x P^{k-t}_y F \|_\infty$$

$$\le \|F - P^k_x F\|_\infty + \|F - P^k_y F\|_\infty + \sum_{t=0}^{k} \|(I-P^t_x)[F-P^{k-t}_y F]\|_\infty + \sum_{t=0}^{k-1} \|(I-P^t_x)[F-P^{k-1-t}_y F]\|_\infty$$

$$\le h^{2r-1} K \{ \|D^{2r}_1 F\|_\infty + \|D^{2r}_2 F\|_\infty + (2k+1) \|D^r_1 D^r_2 F\|_\infty \} .$$

So we have proved the following

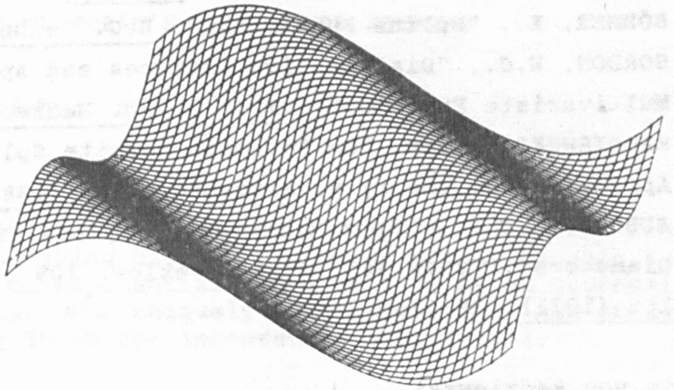
<u>Theorem 2:</u> *For* $F \in C^{2r}_{1,1}(\mathbb{R}^2)$ *the error of reduced Hermite spline interpolation yields*

$$\|F - \bigoplus_{t=0}^{k} P^t_x P^{k-t}_y F\|_\infty = O(h^{2r-1}) .$$

<u>Remark:</u> In the case <u>r = 2</u> the Boolean sum consists only of the spline projectors P^0_x, P^0_y, and the piecewise Hermite interpolation projectors P^{r-1}_x, P^{r-1}_y. For both we can improve Proposition 2 to h^{v+1} instead of $h^{v+1/2}$, and *get* $O(h^{2r})$ *in Theorem* 2.

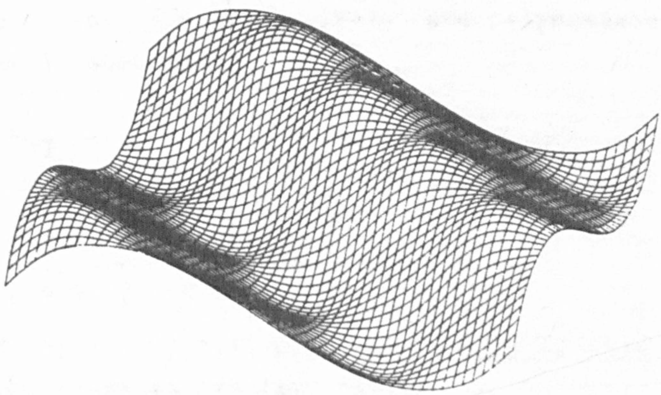
5. Illustrations. For illustration let us consider the interpolation of $F(x,y) := B_3^{o}(x+y) \in C^{1}_{1,1}(\mathbb{R}^2)$, with $N = 5$ and $k = 1$, where B_3^{o} denotes the third Bernoulli function (ploted on [0, 1] × [0, 1]).

F :

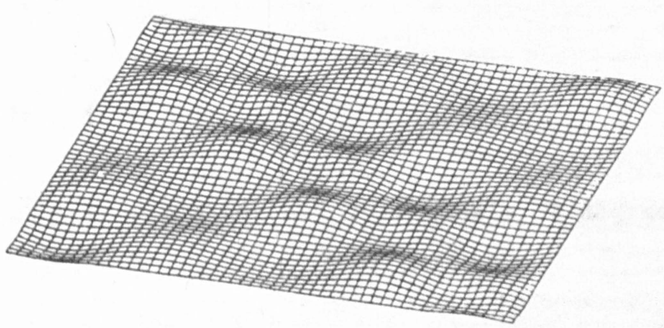


As $D^2 B_3^{o} = 3!B_1^{o}$ has jumps in $x \in \mathbb{Z}$, $F \notin C^{1,1}(\mathbb{R}^2)$, and therefore the Hermite spline tensor product projector $P_x^1 P_y^1$ can not be used, but the boolean projector $P_x^0 P_y^1 + P_x^1 P_y^0 - P_x^0 P_y^0$ is applicable , for example with r = 3 :

$\bigoplus_{t=0}^{1} P_x^{t} P_y^{1-t} F$:



$F - \bigoplus_{t=0}^{1} P_x^{t} P_y^{1-t} F$:



References

1. BASZENSKI,G.;DELVOS,F.-J.;POSDORF,H.; "Boolean Methods in Bivariate Reduced Hermite Interpolation",ISNM 51, (1979) 11-29.
2. BÖHMER, K., "Spline Funktionen" B.G. Teubner Stuttgart, Germany.
3. GORDON, W.J., "Distributive Lattices and Approximation of Multivariate Functions". Proc. Symp. Madison (Wisc.)(1969),223-277.
4. RADZIEWSKI, K.von, "On Periodic Hermite Spline Interpolation", Approximation Theory V, Academic Press,(1986), 531-534.
5. SUBBOTIN, J.N., "Approximation by Spline Functions and Estimates of Diameters", Trudy Mat. Inst. Steklov. 109 (1971) = Proc. Inst. Math. 109 (1971), 39-67.

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