CONSTRUCTIVE THEORY OF FUNCTIONS'87

SOFIA, 1988

AN ANALOGUE OF TURÁN'S QUADRATURE FORMULA

Q.I. Rahman and G. Schmeisser

Abstract. We consider quadrature formulae on the whole real line involving derivatives and having highest degree of precision with respect to entire functions of exponential type. We show by an extremal property that such formulae are uniquely determined. They are analogous to formulae introduced by Turán for integration over [-1,1].

1. The formulae of Turán. Let $X = (x_1, x_2, \ldots, x_n)$ be a system of n distinct points on the real line and let k be a positive integer. By Hermite interpolation every polynomial $P(x)$ of degree less than kn may be represented as

$$(1) \qquad P(x) = \sum_{\nu=1}^{n} \sum_{j=o}^{k-1} P^{(j)}(x_\nu)\, \ell_{\nu j}(x)\,.$$

where $\ell_{\nu j}(x)$ $(\nu = 1,2,\ldots,n;\ j = o,1,\ldots,k-1)$ are polynomials each of degree at most $kn-1$ such that

$$\ell_{\nu m}^{(j)}(x_\mu) = \delta_{\nu\mu}\,\delta_{mj}$$

with

$$\delta_{\rho\lambda} = \begin{cases} o & \text{if } \rho \neq \lambda \\ 1 & \text{if } \rho = \lambda\,. \end{cases}$$

Integrating both sides of (1) over [-1,1] we obtain a special case of the quadrature formulae of L. Tschakaloff [8], namely

$$(2) \qquad \int_{-1}^{1} P(x)\,dx = \sum_{\nu=1}^{n} \sum_{j=o}^{k-1} \lambda_{\nu j} P^{(j)}(x_\nu)$$

with

$$\lambda_{\nu j} = \int_{-1}^{1} \ell_{\nu j}(x)\,dx \qquad (1 \le \nu \le n; \quad o \le j \le k-1).$$

Coming from (1) this formula is valid for all polynomials of degree at most $kn-1$. Nothing more can be said in general. However, in view of Gauss' formula one may ask if by appropriate choice of the system X one can achieve that (2) holds for polynomials up to a degree $m \geq kn$. The answer was given by P. Turán [9] and may be stated as

Theorem A. (i) For even $k \in \mathbb{N}$ there is no real system $X = (x_1, x_2, \ldots, x_n)$ such that (2) holds for all polynomials of degree $m \leq kn$.
(ii) For odd $k \in \mathbb{N}$ there is a unique system $X^* = (x_1^*, x_2^*, \ldots, x_n^*)$ such that (2) holds for all polynomials of degree $m \leq (k+1)n-1$.
(iii) The system X^* is characterized by the fact that

$$\pi_{X^*}(z) := \prod_{\nu=1}^{n} (z - x_\nu^*)$$

minimizes the integral

$$(3) \qquad I_k(P) := \int_{-1}^{1} |P(x)|^{k+1} dx$$

amongst all monic polynomials $P(x)$ of degree n.

The case $k=1$ leads to the classical Gauss formula.

2. Hermite interpolation by entire functions of exponential type. It is known that an entire function of exponential type less than $k\sigma$ is completely determined by its values and those of its derivatives of order at most $k-1$ at the points $\nu\pi/\sigma$ or somewhat displaced points λ_ν, $\nu \in \{0, \pm 1, \pm 2, \ldots\} =: \mathbb{Z}$ (see [3] and [6, Theorem 4]). Such sequences $\Lambda = (\lambda_\nu)_{\nu \in \mathbb{Z}}$ are candidates for systems taking the role of X in the case of Hermite interpolation by entire functions of exponential type. We find it convenient and reasonable to restrict ourselves to sequences described in the following

Definition 1. Let $k \in \mathbb{N}$ and $\sigma > 0$. By $USL^k(\sigma)$ we denote the set of all increasing sequences $\Lambda = (\lambda_\nu)_{\nu \in \mathbb{Z}}$ of real numbers with the following properties:
(i) $\lambda_0 = 0$ and $\lambda_\nu \to \pm\infty$ as $\nu \to \pm\infty$.
(ii) If an entire function f of exponential type less than $k\sigma$ vanishes along with its derivatives of order at most $k-1$ at the points λ_ν, then $f \equiv 0$.
(iii) There exists an entire function ψ_Λ of order 1 and type σ such that $\psi_\Lambda(\mathbb{R}) \subseteq \mathbb{R}$, $\psi_\Lambda'(0) = 1$ and $\psi_\Lambda(\lambda_\nu) = 0$ for $\nu \in \mathbb{Z}$.

(iv) $$\int_{-\infty}^{\infty} x^{-2}|\psi_\Lambda(x)|^{k+1}dx < \infty.$$

In the notation of [7] an element belonging to $USL^k(\sigma)$ is called a k <u>uniqueness sequence of type</u> σ <u>and class</u> $(1,\sigma)$ <u>having the modified</u> L^{k+1} <u>property</u>.

Let us give a few explanations concerning (i) - (iv). Condition (i) means that Λ does not have an accumulation point on $\mathbb{R}$ and $\lambda_o=o$ is a normalization achieved by a translation on $\mathbb{R}$. Obviously (ii) guarantees uniqueness of Hermite interpolation. Requirement (iii) corresponds to the fact that for every system $X = (x_1,x_2,\ldots,x_n)$ of real points there exists a real polynomial $\pi_X(z)$ of degree n vanishing for $z = x_\nu$ $(\nu = 1,2,\ldots,n)$. Of course the integral (3) exists for $P=\pi_X$. An analogous requirement for ψ_Λ and integration over $\mathbb{R}$ turns out to be too strong. Fortunately the weaker condition (iv) is enough for our purpose.

The set $USL^k(\sigma)$ is non-trivial. It contains the sequence $\Lambda^* := (\nu\pi/\sigma)_{\nu\in\mathbb{Z}}$ with $\psi_{\Lambda^*}(x) := \frac{\sin \sigma z}{\sigma}$ and sequences obtained by displacing the points of Λ^* slightly, keeping $\lambda_o=o$ fixed. In many cases (see [4] and [6, p. 1o35-1o36]) there exists a Hermite interpolation formula

(4) $$f(z) = \sum_{\nu=-\infty}^{\infty}\sum_{j=o}^{k-1} f^{(j)}(\lambda_\nu)L_{\nu j}(z)$$

with

$$L_{\nu m}^{(j)}(\lambda_\mu) = \delta_{\nu\mu}\,\delta_{mj}$$

valid for entire functions f of exponential type less than $k\sigma$ under side conditions on their growth on $\mathbb{R}$.

<u>3. Derivation of quadrature formulae</u>. Once a representation (4) holds and $f\in L^1(\mathbb{R})$ we may think of integrating both sides over $\mathbb{R}$. Provided term by term integration is justified we obtain a quadrature formula of the form

(5) $$\int_{-\infty}^{\infty} f(x)dx = \sum_{\nu=-\infty}^{\infty}\sum_{j=o}^{k-1} w_{\nu j}f^{(j)}(\lambda_\nu).$$

Since we started with (4) the exponential type of f has to be less than $k\sigma$, in general. However, in view of Turán's result we may ask if by an appropriate choice of the sequence Λ it is possible to relax the restriction on the type. The answer is "yes" for odd k. The simplest and most natural sequence, namely $(\nu\pi/\sigma)_{\nu\in\mathbb{Z}}$ is a right choice. In fact, in this special case formula (5) was discovered by Kress [4] and further investigated by Olivier and Rahman [5] who obtained

Theorem B. *Let* k *be an odd positive integer and let* $\sigma > o$. *Define*

$$\psi(z) := \prod_{j=1}^{(k-1)/2} \left(1 + \frac{z^2}{j^2}\right) ,$$

$a_{oo} := 1$ *and*

$$a_{j,k-1} := \frac{-1}{j!}\, \psi^{(j)}(o) \quad \textit{for} \quad o \leq j \leq k-1 .$$

If f *is an entire function of exponential type* $(k+1)\sigma$ *belonging to* $L^1(\mathbb{R})$, *then*

$$(6) \qquad \int_{-\infty}^{\infty} f(x)\,dx = \frac{\pi}{\sigma} \sum_{\nu=-\infty}^{\infty} \sum_{\substack{j=o \\ j \text{ even}}}^{k-1} (2\sigma)^{-j} a_{j,k-1} f^{(j)}\left(\frac{\nu\pi}{\sigma}\right) .$$

From now on we shall call every formula (5) a quadrature formula *associated* with $\Lambda = (\lambda_\nu)_{\nu\in\mathbb{Z}}$ no matter whether (5) was produced by integrating (4) or in any other way. For convenient notation we introduce

Definition 2. A sequence $\Lambda \in USL^k(\sigma)$ is said to have the *Turán property* if there exists an associated formula (5), called a *Turán formula*, which is valid for all entire functions of exponential type $(k+1)\sigma$ belonging to $L^1(\mathbb{R})$.

In view of Theorem B we may ask: Are there sequences $\Lambda \in USL^k(\sigma)$ other than $(\nu\pi/\sigma)_{\nu\in\mathbb{Z}}$ which have the Turán property and is the associated Turán formula unique? The answer is given in Section 5 below.

4. Lemmas. We need a few auxiliary results. As an immediate consequence of Hölder's inequaltiy we have

Lemma 1. *Let* $k \in \mathbb{N}$ *and let* f, g *be functions defined on* $\mathbb{R}$ *such that* $f^k(x)/x^2$ *and* $g^k(x)/x^2$ *belong to* $L^1(\mathbb{R})$. *Then for all non-negative integers* ν,μ *with* $\nu+\mu = k$ *the function* $f^\nu(x)g^\mu(x)/x^2$ *also belongs to* $L^1(\mathbb{R})$.

Lemma 2. *Let* k *be an even positive integer and let* a,b *be real numbers with* $b \neq o$. *Then*

$$(a+b)^k > a^k + ka^{k-1}b \quad .$$

Proof. If $a=o$ or if a and b are of the same sign the inequality is obvious. If a and b are of opposite sign and $|a| > |b|$ the desired result is a consequence of Bernoulli's inequality. In all other cases the left hand side is non-negative whereas the right hand side is negative.

Lemma 3. Let $k \in \mathbb{N}$ be odd and let $\Lambda \in USL^k(\sigma)$ have the Turán property. If f is an entire function of exponential type σ such that $f(o) = o$, $f'(o) = 1$ and $f \neq \psi_\Lambda$, then

$$(7) \qquad \int_{-\infty}^{\infty} x^{-2}|\psi_\Lambda(x)|^{k+1}dx < \int_{-\infty}^{\infty} x^{-2}|f(x)|^{k+1}dx \ .$$

Proof. We may assume that the right hand side of (7) is finite because otherwise (7) is trivial.

Suppose first that $f(x)$ is real for real x. For short notation we write $\psi := \psi_\Lambda$ and consider

$$h(z) := f(z) - \psi(z).$$

Clearly, h is of exponential type σ, further $h(x) \not\equiv o$ and

$$(*) \qquad h(o) = h'(o) = o.$$

By Lemma 1 the function $f^\nu(x)\psi^\mu(x)/x^2$ and consequently also $h^\nu(x)\psi^\mu(x)/x^2$ belongs to $L^1(\mathbb{R})$ for all non-negative integers ν,μ with $\nu + \mu = k+1$. Hence

$$(8) \qquad F : x \mapsto \frac{(\psi(x)+h(x))^{k+1}}{x^2} - \frac{\psi^{k+1}}{x^2} - \frac{(k+1)\psi^k(x)h(x)}{x^2}$$

is an entire function of exponential type $(k+1)\sigma$, real for real x and belonging to $L^1(\mathbb{R})$. By Lemma 2 $F(x) > o$ almost everywhere on $\mathbb{R}$ and so

$$(9) \qquad \int_{-\infty}^{\infty} F(x)dx > o \ .$$

Next we note that $\psi^k(x)h(x)/x^2$ itself is an entire function of exponential type $(k+1)\sigma$ belonging to $L^1(\mathbb{R})$. Since by assumption Λ has the Turán property there is an associated Turán formula (5) which in view of $(*)$ yields

$$(1o) \qquad \int_{-\infty}^{\infty} x^{-2}\psi^k(x)h(x)dx = o.$$

Now the desired inequality (7) follows from (8) - (1o).

If $f(z) = \sum_{\nu=o}^{\infty} a_\nu z^\nu$ has complex coefficients we apply the above reasoning first to $g(z) := \sum_{\nu=o}^{\infty} \operatorname{Re} a_\nu z^\nu$ and find

$$\int_{-\infty}^{\infty} x^{-2}|\psi_\Lambda(x)|^{k+1}dx < \int_{-\infty}^{\infty} x^{-2}|g(x)|^{k+1}dx \le \int_{-\infty}^{\infty} x^{-2}|f(x)|^{k+1}dx.$$

Lemma 4. *If $k \in \mathbb{N}$ is odd and $\Lambda \in USL^k(\sigma)$ has the Turán property, then the function ψ_Λ is uniquely determined. All its zeros are simple and are elements of Λ.*

Proof. The uniqueness of ψ_Λ is an immediate consequence of Lemma 3. Suppose $\lambda \in \mathbb{R}$ is a multiple zero of ψ_Λ or a zero not appearing in the sequence Λ. Then

$$\tilde{\psi}_\Lambda(z) := \frac{\lambda}{z-\lambda}\, \psi_\Lambda(z)$$

also satisfies the requirements (iii) and (iv) of Definition 1. This contradicts the uniqueness of ψ_Λ. A similar argument is used to exclude any pair of complex, conjugate zeros.

Lemma 5. *For the sequence $(\nu\pi/\sigma)_{\nu\in\mathbb{Z}}$ formula (6) is the only Turán formula.*

Proof. Suppose there is another such formula. Then the functions

$$f_{\nu\mu} : x \mapsto \sin^\mu(\sigma x) \left(\frac{\sin(\sigma x - \times\pi)}{\sigma x - \nu\pi}\right)^2$$

$$(\mu = o,1,\ldots,k-1; \nu \in \mathbb{Z})$$

are admissible for both of them. Applying these formulae to $f_{k-1,\nu}$ we see that their coefficient of $f_{k-1,\nu}^{(k-1)}(\nu\frac{\pi}{\sigma})$ must be the same. Repeating the argument with $f_{k-2,\nu}, f_{k-3,\nu}, \ldots, f_{o,\nu}$ we successively find that all corresponding coefficients coincide. Coefficients not appearing in (6) explicitly (j odd) are interpreted as being zero.

5. Uniqueness and characterization of the Turán formula. We now come to our main result analogous to Theorem A.

Theorem 1. (i) *For even $k \in \mathbb{N}$ there is no element $\Lambda \in USL^k(\sigma)$ and no $\varepsilon > o$ such that (5) is valid for all entire functions of expo-*

nential type $k\sigma + \varepsilon$ belonging to $L^1(\mathbb{R})$.
(ii) For odd $k \in \mathbb{N}$ the sequence $\Lambda^* := (\nu\pi/\sigma)_{\nu\in\mathbb{Z}}$ is the only element of $USL^k(\sigma)$ which has the Turán property and (6) is the only Turán formula associated with it.
(iii) The sequence Λ^* is characterized by the fact that ψ_{Λ^*} minimizes the integral

$$I_k(f) := \int_{-\infty}^{\infty} x^{-2} |f(x)|^{k+1} dx$$

amongst all entire functions of exponential type σ such that $f(o)=o$, $f'(o)=1$.

Proof. Let k be even. Since an entire function of exponential type belonging to $L^1(\mathbb{R})$ is bounded on $\mathbb{R}$ (see [1, Theorem 6.7.15]) we deduce from property (iv) in Definition 1 that the entire function

$$f := z \mapsto \psi_\Lambda^k(z) \left(\frac{\sin(\varepsilon z/4)}{z}\right)^4$$

of exponential type $k\sigma + \varepsilon$ belongs to $L^1(\mathbb{R})$. Furthermore, $f(x) > o$ almost everywhere on $\mathbb{R}$. Hence for this function the left hand side of (5) is positive whereas the right hand side vanishes. This proves (i).

For odd k we know from Theorem B that Λ^* has the Turán property and ψ_{Λ^*} may be taken as $(\sin \sigma z)/\sigma$. By Lemmas 3 and 4 $\psi_{\Lambda^*}(z) \equiv \equiv \psi_\Lambda(z) \equiv (\sin \sigma z)/\sigma$ for every $\Lambda \in USL^k(\sigma)$ having also the Turán property. Now it follows from Lemma 4 that $\Lambda = \Lambda^*$. Finally Lemma 5 tells us that (6) is the only Turán formula associated with Λ^*. This proves (ii). The statement (iii) is an immediate consequence of Lemmas 3 and 4.

Since for $\psi_{\Lambda^*}(z) \equiv (\sin \sigma z)/\sigma$ the integral on the left hand side of (7) can be calculated explicitly (see [2], p. 446) we deduce from Lemma 3 the following inequality which may be of independent interest.

Theorem 2. Let k be an odd positive integer. Then for every entire function f of exponential type σ

$$|f'(o)|^{k+1} \le \frac{(2\sigma)^k}{2\pi} \frac{\left(\left(\frac{k-1}{2}\right)!\right)^2}{(k-1)!} \int_{-\infty}^{\infty} x^{-2} |f(x) - f(o)|^{k+1} dx.$$

Equality is attained if and only if $f(z) = a+b\cdot\sin \sigma z$ where $a,b \in \mathbb{C}$.

Remark. The requirement (ii) of Definition 1 (uniqueness) was only used to introduce the quadrature formula (5) in analogy with Turán's approach. In none of our lemmas and theorems (ii) was really employed. In fact, all our results are also true with $USL^k(\sigma)$ replaced by $SL^k(\sigma)$ where the latter set consists of all sequences satisfying only (i), (iii) and (iv) of Definition 1.

Example. The sequence $\Lambda^* = (\nu\pi/\sigma)_{\nu\in\mathbb{Z}}$ does not belong to $USL^k(\frac{k+1}{k}\sigma)$ since $f(z) := (\sin \sigma z)^k$ is a counter-example for (ii). However $\Lambda^* \in SL^k(\frac{k+1}{k}\sigma)$ where $\psi_{\Lambda^*}(z)$ may be taken as $\frac{1}{\sigma}\sin(\sigma z)\cos(\frac{\sigma}{k}z)$. Of course, as an element of this latter set Λ^* does not have the Turán property.

References

1. R.P. Boas. Entire functions. Academic Press, New York, 1954.

2. I.S. Gradshteyn and I.M. Ryzhik. Tables of integrals, series, and products. Academic Press, New York, 1980.

3. J. Korevaar. Functions of exponential type bounded on a sequence of points. Ann. Polon. Math. 22 (1949-1950), 2o7-234.

4. R. Kress. On the general Hermite cardinal interpolation. Math. Comp. 26 (1972), 925-933.

5. P. Olivier et Q.I. Rahman. Sur une formule de quadrature pour des fonctions entières. Modélisation mathématique et Analyse numérique 20 (1986), 517-537.

6. Q.I. Rahman. Interpolation of entire functions. Amer. J. Math. 87 (1965), 1o29-1o76.

7. Q.I. Rahman and G. Schmeisser. On a Gaussian quadrature formula for entire functions of exponential type. (To appear in: International Series of Numerical Mathematics, Birkhäuser).

8. L. Tschakaloff. Über eine allgemeine Quadraturformel. Comptes Rendus de l'Acad. Bulgare des Sciences 1 (1948), 9-12.

9. P. Turán. On the theory of the mechanical quadrature. Acta Scient. Math. 12 (195o), 3o-37.

Dept. de Math. et de Statistique
Université de Montréal
Montréal, P.Q. H3C 3J7
CANADA

Mathematisches Institut
Universität Erlangen - Nürnberg
D - 8520 Erlangen
GERMANY (FED. REP.)