BEST CONSTANTS FOR THE SIMULTANEOUS APPROXIMATION OF PERIODIC FUNCTIONS BY TRIGONOMETRIC POLYNOMIALS

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Let $C^r_{2\pi}$ the r times differentiable 2π periodic functions then we get

$$f(x) = \int_0^{2\pi} f^{(r)}(x-t)D_r(t)dt + \frac{1}{2\pi}\int_0^{2\pi} f(t)dt$$

with $D'_{s+1} = D_s$, $\frac{1}{2\pi}\int_0^{2\pi} D_s(t)dt = 0$ $\qquad (1 \le r \le s,\ r, s \in \mathbb{N})$

$$D_s(t) = \frac{1}{\pi}\sum_{k=1}^{\infty} \frac{1}{k^s}\cos(kt - \frac{\pi}{2}s) \qquad (D_1(t) := \frac{\pi - t}{\pi},\ 0<t<2\pi).$$

Except factor D_s is the s^{th} polynomial of Bernoulli.

A trigonometric polynomial of approximation of order n for f can be written in the form

$$s_n(x) = \int_0^{2\pi} f^{(r)}(x-t)T_{rn}(t)dt + \frac{1}{2\pi}\int_0^{2\pi} f(t)dt.$$

The deviation of the best polynomial of approximation s^*_n can be estimated in the following form

$$\|f - s^*_n\|_\infty \le \int_0^{2\pi} |D_r(t) \quad T^*_{rn}(t)||f^{(r)}(x-t)|dt$$

$$\le \|f^{(r)}\|_\infty \int_0^{2\pi} |D_r(t) - T^*_{rn}(t)|dt.$$

Now we have a problem of L_1 - approximation for the function D_r:

$$\int_0^{2\pi} |D_r - T^*_{rn}| = K_r \frac{1}{(n+1)^r}$$

and we get the result of Favard, Achieser, Krein ([3], [1]):

Let $W_r := \{f \in C_{2\pi} \mid f^{(r-1)}$ absol.contin., $|f^{(r)}| \le 1$ a.e.$\}$ and

$E_n(W_r) := \sup\limits_{f \in W_r} E_n(f)$ we have

Theorem. 1. (Favard, Achieser, Krein)

$$(*) \quad E_n(W_r) = K_r \frac{1}{(n+1)^r}$$

$$\frac{\pi^2}{8} = K_2 < K_4 < \ldots < \frac{4}{\pi} < \ldots < K_1 = \frac{\pi}{2}$$

(*) ist best possible.

Corollary. $E_n(f) \le \dfrac{K_r}{(n+1)^r} \, ||f^{(r)}||_\infty$ for $f \in C^r_{2\pi}$

The best polynomial T^*_{rn} for D_r in the L_1-norm is a interpolating polynomial in equidistant nodes.

It is also possible to evaluate these polynomials. We have the

Proposition.

$$T^*_{rn}(t) = \frac{1}{\pi} \left({}_r\gamma_0^{(n)} + \left(\frac{\pi}{2n}\right)^r \sum_{k=1}^{n-1} {}_r\gamma_k^{(n)} \cos\left(kt - r\frac{\pi}{2}\right) \right)$$

with

$${}_r\gamma_k^{(n)} := \begin{cases} \dfrac{1}{(r-1)!} \sec^{(r-1)}\left(\dfrac{(n-k)\pi}{2n}\right) & r \text{ even} \\[2ex] \dfrac{1}{(r-1)!} \tan^{(r-1)}\left(\dfrac{(n-k)\pi}{2n}\right) & r \text{ odd} \end{cases}$$

$$ {}_r\gamma_0^{(n)} := \begin{cases} \dfrac{(-1)^{r/2}}{(2n)^r} \sum_{j=1}^{\infty} \dfrac{(-1)^j}{j^r} & r \text{ even} \\ 0 & r \text{ odd}. \end{cases} \qquad ([5], [6]) $$

For the case of simultaneous approximation we have analogously for $f \in C_{2\pi}^r$, $0 \le p \le r-1$ and a polynomial of approximation

$$ f^{(p)}(x) = \int_0^{2\pi} f^{(r)}(x-t) D_{r-p}(t)\, dt $$

$$ s_n^{(p)}(x) = \int_0^{2\pi} f^{(r)}(x-t) T_{rn}^{(p)}(t)\, dt $$

and the deviation

$$ \|f^{(p)} - s_n^{(p)}\|_\infty \le \int_0^{2\pi} |f^{(r)}(x-t)|\, |D_{r-p}(t) - T_{rn}^{(p)}(t)|\, dt. $$

If $f \in W_r$ and if we choose the polynomial $T_{rn}^*(t)$ which is minimizing D_r in the L_1-norm we get for the corresponding polynomial of approximation s_n

$$ \|f^{(p)} - s_n^{(p)}\|_\infty \le \int_0^{2\pi} |D_{r-p}(t) - T_{rn}^{*(p)}(t)|\, dt $$

$$ = K_r^{(p)}(n) \frac{1}{(n+1)^{r-p}} \qquad \text{with } K_r^{(0)}(n) = K_r \quad \text{(from Th.1)} $$

and we have

<u>Theorem 2.</u> (Simultaneous approximation) For every $f \in W_r$ there exist a polynomial of approximation s_n with

$$ \|f - s_n\|_\infty \le K_r \frac{1}{(n+1)^r} \qquad (K_r \text{ from Th.1}) $$

$$ \|f^{(p)} - s_n^{(p)}\|_\infty \le K_r^{(p)}(n) \frac{1}{(n+1)^{r-p}} \le K_r^{(p)} \frac{1}{(n+1)^{r-p}} \qquad (0 \le p \le r-1,\ p, r \in \mathbb{N}), $$

the constants are defined by $\|D_r - T_{nr}^*\|_1 (n+1)^r =: K_r$,

$$ \|D_{r-p} - T_{nr}^{*(p)}\|_1 (n+1)^{r-p} =: K_r^{(p)}(n), \quad K_r^{(p)} := \sup_n K_r^{(p)}(n). $$

For the extremal functions of Th.1 we have b.a. polynomials and "=" instead of "≤".

Because it was not possible for us to determine the constants $K_r^{(p)}(n)$ and $K_r^{(p)}$ by analytical methods, we calculated these constants numerically. For reason of the factors $(n+1)^{r-p}$ we could calculate the constants for large r-p only for small n, although we have computated with fourfold precision. Instead of $K_r^{(p)}$ we have $K_{rN}^{(p)} := \sup_{1\le n\le N} K_r^{(p)}(n)$. We have computed the following constants.

r	(N=104) $K_{rN}^{(r-1)}$	(N=84) $K_{rN}^{(r-2)}$	(N=50) $K_{rN}^{(r-3)}$	(N=40) $K_{rN}^{(r-4)}$	(N=30) $K_{rN}^{(r-5)}$	(N=24) $K_{rN}^{(r-6)}$
1	1.571	-	-	-	-	-
2	1.603	1.234	-	-	-	-
3	1.719	1.326	1.292	-	-	-
4	1.792	1.386	1.313	1.268	-	-
5	1.865	1.450	1.357	1.290	1.275	-
6	1.923	1.502	1.397	1.319	1.286	1.273
7	1.975	1.549	1.436	1.351	1.308	1.281
8	2.021	1.590	1.471	1.381	1.331	1.297
9	2.062	1.628	1.504	1.410	1.355	1.316
10	2.099	1.662	1.534	1.437	1.378	1.335
12	2.164	1.722	1.588	1.485	1.420	1.371
14	2.219	1.774	1.634	1.527	1.457	1.403
16	2.268	1.819	1.675	1.564	1.490	1.432
18	2.311	1.860	1.711	1.597	1.519	1.457
20	2.349	1.896	1.743	1.627	1.544	1.478
22	2.384	1.929	1.772	1.653	1.566	1.497
24	2.416	1.959	1.798	1.676	1.586	1.512
26	2.445	1.986	1.822	1.697	1.603	1.526

r	(N=18) $K_{rN}^{(r-7)}$	(N=14) $K_{rN}^{(r-8)}$	(N=14) $K_{rN}^{(r-9)}$	(N=12) $K_{rN}^{(r-10)}$	(N=8) $K_{rN}^{(r-12)}$	(N=6) $K_{rN}^{(r-14)}$
7	1.273	-	-	-	-	-
8	1.297	1.273	-	-	-	-
9	1.291	1.278	1.273	-	-	-
10	1.306	1.286	1.277	1.273	-	-
12	1.334	1.308	1.292	1.281	1.273	-
14	1.361	1.329	1.310	1.294	1.277	1.273
16	1.384	1.346	1.326	1.306	1.282	1.274
18	1.403	1.360	1.340	1.317	1.286	1.276
20	1.420	1.372	1.351	1.325	1.288	1.276
22	1.433	1.381	1.359	1.331	1.290	1.277
24	1.444	1.388	1.366	1.336	1.291	1.277
26	1.453	1.393	1.371	1.339	1.292	1.277

Tendencies

A) $K_r^{(0)} = K_r \xrightarrow[r\to\infty]{} \frac{4}{\pi} = 1.27324$

$K_2^{(1)} = 1.603 > K_3^{(1)} > \ldots > K_{12}^{(1)} = 1.275 \to \frac{4}{\pi}$

$K_3^{(2)} = 1.719 > K_4^{(2)} > \ldots > K_{12}^{(2)} = 1.281 \to \frac{4}{\pi}$

$K_4^{(3)} = 1.792 > K_5^{(3)} > \ldots > K_{12}^{(3)} = 1.292 \to \frac{4}{\pi}$

B) $K_r^{(r-1)}$ (1.column), $K_r^{(r-2)}$ (2.column),..., monoton increasing with r

C) $K_r^{(j)}(n)$ monoton increasing with n to $K_r^{(j)}$

Example:

n	7	16	32	64	128	256	512
$K_4^{(1)}(n)$	1.31114	1.31278	1.31309	1.31317	1.31319	1.31320	1.31320
$K_4^{(2)}(n)$	1.37637	1.38382	1.38523	1.38559	1.38568	1.38570	1.38571
$K_4^{(3)}(n)$	1.77237	1.78810	1.79110	1.79186	1.79206	1.79210	1.79212

Relations to the papers of Czipszer/Freud [2] and Garcavi [4] for simultaneous approximation of periodic functions:

For $f \in W^{(r)}$ (class of 2π-periodic functions with bounded r-th derivative) let $E_n(f^{(s)}) = \inf_{t_n} ||f^{(s)} - t_n||_\infty$.

Then there exist approximating polynomials $t_{rn}(f)$ with

$$||f^{(s)} - t_{rn}^{(s)}(f)|| \le C_{nr}\, E_n(f^{(s)}) , \quad s \in \{0,\ldots r\}$$

and

$$C_{n,r} := \sup_{f\in W^{(r)}} C_{n,r}(f), \qquad C_{n,r}(f) := \inf_{t_n} \max_{0\le s\le r} \frac{||f^{(s)} - t_n^{(s)}(f)||}{E_n(f^{(s)})} .$$

27

Garcavi and Czipszer/Freud showed

$$C_{n,r} \le \frac{4}{\pi^2} \ln(p+1) + O(1), \quad p = \min(n,r) \qquad (O(1) \le \pi e + 4)$$

The factor $4/\pi^2$, due to Garcavi, is best possible.
The results were obtained with the help of generalized sums of Vallée-Poussin.
Czipszer/Freud were interested first in the following problem: If $||f - P_n||_\infty < \varepsilon$ is known, what can one say about $||f^{(s)} - P_n^{(s)}||_\infty$?
Now we compare the differences $K_{r+1}^{(r)} - K_r^{(r-1)}$ with the analogous expressions due to Garcavi:

$$(C_{n,r+1} - C_{n,r}) \frac{4}{\pi} \sim \frac{4}{\pi^2} (\ln(r+2) - \ln(r+1)) \frac{4}{\pi}$$

$(E_n(W_r) = K_r/(n+1)^r$, $K_r \to 4/\pi$ for $r \to \infty)$.

r	2	3	4	5	6	7
$K_{r+1}^{(r)} - K_r^{(r-1)}$	0,116	0,073	0,073	0,058	0,052	0,046
$4^2/\pi^3(\ln(r+2)-\ln(r+1))$	0,148	0,115	0,094	0,080	0,069	0,061

r	8	9	11	15	19	23
$K_{r+1}^{(r)} - K_r^{(r-1)}$	0,041	0,037	0.033	0,025	0,020	0,017
$4^2/\pi^3(\ln(r+2)-\ln(r+1))$	0,054	0.049	0.041	0,031	0,025	0.021

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