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2-PERIODIC LACUNARY TRIGONOMETRIC INTERPOLATION: the $(0;M)$ Case.

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1. Introduction. For a given positive integer m, define the equidistant *nodes* $x_k := k\pi/m$ $(k = 0,1,\ldots,2m-1)$ in $[0,2\pi)$, and let $T_{m,\varepsilon}$ (where $\varepsilon = 0$ or $\varepsilon = 1$) denote the space of all trigonometric polynomials $t_m(x)$, of the form

$$t_m(x) = a_0 + \sum_{k=1}^{m-1} (a_k \cos kx + b_k \sin kx) + a_m \cos\left(mx + \frac{\varepsilon\pi}{2}\right), \tag{1.1}$$

where the coefficients $\{a_k\}_{k=0}^{m}$ and $\{b_k\}_{k=1}^{m-1}$ are complex numbers. It is evident that the dimension of the linear space $T_{m,\varepsilon}$ is $2m$.

We consider the following interpolation problem. First, break the set of nodes $\{x_k\}_{k=0}^{2m-1}$ into the *two* disjoint subsets $\{x_{2k}\}_{k=0}^{m-1}$ and $\{x_{2k+1}\}_{k=0}^{m-1}$. Then, given any positive integer M, the *2-periodic* $(0;M)$ *lacunary trigonometric interpolation problem* (the "$(0;M)$ interpolation problem" for short) is to determine, from $2m$ given arbitrary complex numbers $\{\alpha_\nu\}_{\nu=0}^{m-1}$ and $\{\beta_\nu\}_{\nu=0}^{m-1}$, a *unique* $t_m(x) \in T_{m,\varepsilon}$ such that

$$t_m(x_{2\nu}) = \alpha_\nu, \text{ and } t_m^{(M)}(x_{2\nu+1}) = \beta_\nu \qquad (\nu = 0,1,\ldots,m-1), \tag{1.2}$$

where $t_m^{(M)}(x) := \frac{d^M}{dx^M} t_m(x)$. Our first problem is

> P_1: determine necessary and sufficient conditions on m, M and ε such that there is a *unique* $t_m(x) \in T_{m,\varepsilon}$ satisfying (1.2).

If P_1 is solvable, then the specific polynomials $\{r_{2j}(x)\}_{j=0}^{m-1}$ and $\{r_{2j+1}(x)\}_{j=0}^{m-1}$, called the *fundamental polynomials of the* $(0;M)$ *interpolation problem,* are defined by

$$r_{2j}(x_{2\nu}) = \delta_{j,\nu}, r_{2j}^{(M)}(x_{2\nu+1}) = 0 (\nu = 0,1,\ldots,m \quad 1), \tag{1.3}$$

and

$$r_{2j+1}(x_{2\nu}) = 0, \qquad r_{2j+1}^{(M)}(x_{2\nu+1}) = \delta_{j,\nu} (\nu = 0, 1, \ldots, m-1). \tag{1.4}$$

(These fundamental polynomials then evidently span $T_{m,\varepsilon}$.) This brings us to

P_2 : if P_1 is solvable, find *explicit* representations for the fundamental polynomials (of the $(0; M)$ interpolation problem) of (1.3)-(1.4).

We note that since the elements of $T_{m,\varepsilon}$ are 2π-periodic, then

$$r_{2j}(x) = r_0(x - x_{2j}) \text{ and } r_{2j+1}(x) = r_1(x - x_{2j}) \quad (j = 0, 1, \ldots, m-1). \tag{1.5}$$

Thus, to solve the problem P_2, it suffices to explicitly find only the fundamental polynomials $r_0(x)$ and $r_1(x)$.

If $C_{2\pi}$ and $C_{2\pi}^{(M)}$ respectively denote the sets of continuous, and M-times continuously differentiable, 2π-periodic functions defined on $\mathbb{R}$, then the linear operator $L_n : C_{2\pi}^{(M)} \to T_{M,\varepsilon}$ can be defined, for any $f(x) \in C_{2\pi}^{(M)}$, by

$$(L_n f)(x) := \sum_{j=0}^{m-1} f(x_{2j})\, r_{2j}(x) + \sum_{j=0}^{m-1} f^{(M)}(x_{2j+1})\, r_{2j+1}(x). \tag{1.6}$$

Similarly, the linear operator $\tilde{L}_n : C_{2\pi} \to T_{M,\varepsilon}$ can be defined for any $f(x) \in C_{2\pi}$, by

$$(\tilde{L}_n f)(x) := \sum_{j=0}^{m-1} f(x_{2j})\, r_{2j}(x) + \sum_{j=0}^{m-1} \beta_{2j+1}\, r_{2j+1}(x), \tag{1.7}$$

where the β_{2j+1} depend (in a specified way) on values of $f(x)$ in $0 \le x \le 2\pi$.

Our results are as follows. In Section 2, we give a solution to problem P_1, and in Section 3, we solve the problem P_2 by determining explicitly the associated fundamental polynomials for the $(0; M)$ interpolation problem. Section 4 deals with the operators defined in (1.6) and (1.7).

By way of background, the problem of lacunary trigonometric interpolation on equidistant nodes was initiated by O. Kis [2], in 1960, and this was later extended by Sharma and Varma [4], in 1965. In 1980, Cavaretta, Sharma, and Varga [1] further generalized the problem to $(0, m_1, m_2, \ldots, m_q)$ *lacunary trigonometric interpolation,* defined as follows. Given arbitrary distinct positive integers $m_1, m_2, \ldots, m_q$, and given arbitrary complex numbers $\{\alpha_{k,\nu}\}_{k=0,\nu=0}^{m-1,q}$, find conditions on the $\{m_\nu\}_{\nu=1}^{q}$ such that there is a unique trigonometric polynomial $t(x)$, of type (1.1) of suitable order and suitable $\varepsilon = 0$ or $\varepsilon = 1$, such that (with $x_k := k\pi/m$ $(k = 0, 1, \ldots, 2m-1)$ and $m_0 := 0$)

$$t^{(m_\nu)}(x_k) = \alpha_{k,\nu} \qquad (k = 0, 1, \ldots, m-1\, ;\ \nu = 0, 1, \ldots, q). \tag{1.8}$$

It turns out (cf. Theorems 1 and 2 of [1]) that the solution of this above lacunary trigonometric interpolation problem depends, remarkably, only on the *number* of even and odd integers in $\{m_\nu\}_{\nu=0}^{q}$, and this is a condition which can be very easily checked. (For more on lacunary trigonometric interpolation, see chapter 11 of [3].)

Noting that the derivative conditions, imposed by the given numbers $\{m_\nu\}_{\nu=0}^{q}$, are applied in exactly the same manner at *all* points x_k in (1.8), Smith, Sharma, and Tzimbalario [5] then introduced the notion of *p-periodic lacunary trigonometric interpolation* on mp equidistant nodes ($p \geq 2$), in which the equidistant nodes are broken into p disjoint sets, and interpolation conditions $\{0, m_1^{(j)}, \ldots, m_{q_j}^{(j)}\}$ are then applied on all the points of the j^{th} set of nodes, ($j = 0, 1, \ldots, p-1$). While necessary and sufficient conditions for the unique solvability of this p-periodic lacunary interpolation problem on mp equidistant nodes is given in [5], these conditions depend on the nonvanishing of certain determinants of order p, conditions which are of course more difficult to check. The point of this present study was to examine particular 2-periodic lacunary trigonometric interpolation problems, in the hope that simple, easily checked, necessary and sufficient conditions (for the unique solvability of these problems), might be derived, in the spirit of the results of [1]. As can be seen from Theorem 1, this is indeed the case.

2. The Problem P_1. We shall prove here

Theorem 1.

a) *If M is an odd positive integer, then the $(0; M)$-interpolation problem P_1 on $2m$ equidistant nodes $\{x_k\}_{k=0}^{2m-1}$ is solvable iff $\varepsilon = 1$ and m is odd.*

b) *If M is an even positive integer, then the $(0; M)$-interpolation problem P_1 on $2m$ equidistant nodes $\{x_k\}_{k=0}^{2m-1}$ is solvable iff $\varepsilon = 0$ (and m is arbitrary).*

Proof. We shall consider the *homogeneous* $(0; M)$-interpolation problem, and show that if the trigonometric polynomial $t_m(x)$ from $T_{m,\varepsilon}$ satisfies

$$t_m(x_{2\nu}) = 0, \text{ and } t_m^{(M)}(x_{2\nu+1}) = 0 \qquad (\nu = 0, 1, \ldots, m-1), \tag{2.1}$$

then $t_m(x) \equiv 0$ iff the conditions of Theorem 1 are satisfied.

First, any $t_m(x)$ in $T_{m,\varepsilon}$ can, with $z := e^{ix}$, be expressed as

$$t_m(x) = z^{-m} R_{2m}(z), \tag{2.2}$$

where $R_{2m}(z) := \sum_{j=0}^{2m} c_j z^j$, an element in π_{2m}, satisfies from (1.1) the additional condition

$$(-1)^{\varepsilon+1} c_{2m} + c_0 = 0. \tag{2.3}$$

(Such a condition is necessary since $\dim \pi_{2m} = 2m+1$, while $\dim T_{m,\varepsilon} = 2m$.) Noting that $\frac{d}{dx} = iz\frac{d}{dz}$, we define the differential operator $\theta := iz\frac{d}{dz}$. With $z_\nu := e^{ix_\nu}$ ($\nu = 0, \ldots, 2m-1$), then (2.1) and (2.2) yield

$$R_{2m}(z_{2\nu}) = 0, \text{ and } \theta^M(z^{-m} R_{2m}(z))\big|_{z=z_{2\nu+1}} = 0 \qquad (\nu = 0, 1, \ldots, m \quad 1). \tag{2.4}$$

As the points $z_{2\nu}$ are m-th roots of unity, the first condition of (2.4) implies that

$$R_{2m}(z) = (z^m - 1)\{Q_0(z) + c_{2m} z^m\}, \qquad (Q_0(z) \in \pi_{m-1}), \tag{2.5}$$

and (2.3) then becomes

$$(-1)^{\varepsilon+1}\, c_{2m} - Q_0(0) = 0. \tag{2.6}$$

Next, write $Q_0(z) = \sum_{j=0}^{m-1} d_j\, z^j$. Since $\theta^M z^\lambda = (i\lambda)^M z^\lambda$ for any real λ, and since $z_{2\nu+1}^m = -1$, then the second condition of (2.4) becomes, with (2.5), just

$$\sum_{j=0}^{m-1} d_j\, \{j^M + (j-m)^M\}(z_{2\nu+1}^j) \; - \; c_{2m}\, m^M = 0 \qquad (\nu = 0, 1, \ldots, m-1). \tag{2.7}$$

Now, as the particular polynomial (in π_{m-1}), defined by

$$q(z) := \sum_{j=0}^{m-1} d_j\, \left\{j^M + (j-m)^M\right\} z^j - c_{2m} m^M,$$

necessarily vanishes, from (2.7), in m distinct points, then $q(z)$ is *identically zero,* and its Taylor coefficients must all be zero:

$$\begin{cases} d_0(-m)^M - c_{2m}\, m^M & = 0, \\ d_j(j^M + (j-m)^M) & = 0, \qquad (j = 1, \ldots, m-1). \end{cases} \tag{2.8}$$

Suppose first that M is an *odd* positive integer. In this case, the coefficient of d_j from (2.8) is $j^M + (j-m)^M = j^M - (m-j)^M$, for $1 \le j \le m-1$. As is easily verified, $j^M - (m-j)^M$ is nonzero for all $1 \le j \le m-1$, iff m is odd. Thus, if M is odd and m is odd, then $d_j = 0$ for all $1 \le j \le m-1$. Moreover, if M is odd, then (2.6) and the first equation of (2.8) together give that

$$d_0 \; + \; c_{2m} = 0, \text{ and } (-1)^{\varepsilon+1}\, c_{2m} - d_0 = 0. \tag{2.9}$$

Hence, for $\varepsilon = 1$, (2.9) yields $d_0 = c_{2m} = 0$, and, from (2.5) and (2.2), $t_m(x) \equiv 0$. On the other hand, if M is an odd positive integer, and if m is even or if m is odd and $\varepsilon = 0$, the above argument shows that there is a $t_m(x) \not\equiv 0$ which satisfies the homogeneous interpolation problem (2.1). Thus, when M is odd, the $(0; M)$ interpolation problem on $2m$ equidistant nodes $\{x_k\}_{k=0}^{2m-1}$ is uniquely solvable iff m is odd and $\varepsilon = 1$, which establishes a) of Theorem 1.

Finally, suppose that M is an *even* positive integer. Then, $j^M + (j-m)^M > 0$ for all $1 \le j \le m-1$, and hence $d_j = 0$ for all $1 \le j \le m-1$ from (2.8). Then, (2.6) and the first equation of (2.8) together give that

$$d_0 - c_{2m} = 0, \text{ and } (-1)^{\varepsilon+1} c_{2m} - d_0 = 0, \tag{2.10}$$

and when $\varepsilon = 0$, we get $d_0 = c_{2m} = 0$, so that $t_m(x) \equiv 0$ from (2.5) and (2.2). Thus, when M is even, the $(0; M)$ interpolation problem on $2m$ equidistant nodes $\{x_k\}_{k=0}^{2m-1}$ is uniquely solvable iff $\varepsilon = 0$ and m arbitrary. □

3. The Fundamental Polynomials $r_0(x)$ and $r_1(x)$. We first consider the case of explicitly determining the fundamental polynomials $r_0(x)$ and $r_1(x)$, of $(0;M)$-trigonometric interpolation in the uniformly spaced points $\{x_k\}_{m=0}^{2m-1}$, *when M is odd.* From Theorem 1, we necessarily have m odd and $\varepsilon = 1$.

To begin, consider the fundamental polynomial $r_1(x)$ in $T_{m,1}$ which, from (1.4), satisfies

$$r_1(x_{2\nu}) = 0, \text{ and } r_1^{(M)}(x_{2\nu+1}) = \delta_{0,\nu} \qquad (\nu = 0, 1, \ldots, m-1). \tag{3.1}$$

Since the associated points $z_{2\nu} = e^{ix_{2\nu}}$ are m-th roots of unity, if follows from (2.2) and the first conditions of (3.1) that

$$r_1(x) = z^{-m}(z^m - 1)\, R_1(z) = (1 - z^{-m})\, R_1(z) \qquad (R_1(z) \in \pi_m). \tag{3.2}$$

On writing $R_1(z) = \sum_{j=0}^{m} c_j\, z^j$, then since $r_1(x)$ is an element of $T_{m,1}$ (i.e., $\varepsilon = 1$), it follows (cf. (2.6)) that

$$c_0 - c_m = 0. \tag{3.3}$$

Because (cf. (3.1)) $r_1^{(M)}(x_1) = 1$ and $r_1^{(M)}(x_{2\nu+1}) = 0$ for $\nu = 1, \ldots, m-1$, we require that

$$\theta^M\{(1 - z^{-m})R_1(z)\}|_{z=z_{2\nu+1}} = \begin{cases} 1, & \text{if } \nu = 0; \\ 0, & \text{if } \nu = 1, 2, \ldots, m-1. \end{cases} \tag{3.4}$$

On applying the operator θ^M, and on recalling that $z_{2\nu+1}^m = -1$ and M is odd, the above conditions reduce to

$$i^M \left\{ \sum_{j=0}^{m-1} c_j \left(j^M - (m-j)^M\right) z_{2\nu+1}^j - c_m\, m^M \right\} = \begin{cases} 1, & \text{if } \nu = 0; \\ 0, & \text{if } \nu = 1, 2, \ldots, m-1. \end{cases} \tag{3.5}$$

Because $z_1^m = -1$, the polynomial $s_{m-1}(z)$(in π_{m-1}), defined by

$$\begin{aligned} s_{m-1}(z) &:= -\frac{z_1(z^m+1)}{m(z-z_1)} = -\frac{z_1}{m}\left\{\frac{z^m - z_1^m}{z - z_1}\right\} \\ &= -\frac{\{z_1 z^{m-1} + z_1^2 z^{m-2} + \ldots + z_1^m\}}{m}, \end{aligned} \tag{3.6}$$

is just the Lagrange interpolant, in the points $\{z_{2\nu+1}\}_{\nu=0}^{m-1}$, of the right side of (3.6). Thus, with (3.5), the following two polynomials are identical:

$$i^M \left\{ \sum_{j=0}^{m-1} c_j \left(j^M - (m-j)^M\right) z^j - c_m\, m^M \right\} \equiv s_{m-1}(z). \tag{3.7}$$

On equating the coefficients of z^j on both sides of (3.7), we see from (3.6), after some simplifications, that

$$c_0 + c_m = \frac{-(i^{-M})}{M+1}, \tag{3.8}$$

and

$$c_j = \frac{i^{-M}}{m\,[\,j^M - (m-j)^M]\,z_1^j} \qquad (j = 1, 2, \ldots, m-1). \tag{3.9}$$

Coupling (3.8) with (3.3) then gives

$$c_0 = c_m = \frac{-(i^{-M})}{2m^{M+1}}. \tag{3.10}$$

Returning to $r_1(x)$ of (3.2), we have

$$r_1(x) = (1 - z^{-m}) \sum_{j=0}^{m} c_j\, z^j = \sum_{j=0}^{m} c_j\, z^j - \sum_{j=0}^{m} c_{m-j}\, z^{-j}.$$

Now, from (3.9) and (3.10), it is easy to see that $c_{m-j} = z_1^{-2j} c_j$, for all $0 \le j \le m$, so that

$$r_1(x) = \frac{-(-i)^M}{2m^{M+1}}(z^m - z^{-m}) + \frac{i^{-M}}{m} \sum_{j=1}^{m-1} \frac{1}{[\,j^M - (m-j)^M]} \left\{ \left(\frac{z}{z_1}\right)^j - \left(\frac{z}{z_1}\right)^{-j} \right\}.$$

Thus, as $z := e^{ix}$, the above representation gives

$$r_1(x) = \frac{(-1)^{\frac{M+1}{2}}}{m} \left\{ \frac{\sin mx}{m^M} + 2\sum_{j=1}^{m-1} \frac{\sin[\,j(x - x_1)]}{[\,(m-j)^M - j^M]} \right\} \qquad (M \text{ odd}),$$

which gives an explicit representation for the fundamental polynomial $r_1(x)$ of the $(0; M)$ interpolation problem when M is odd. In a similar fashion (we suppress the details), the fundamental polynomials ($r_0(x)$ (when M is odd), and $r_1(x)$ and $r_0(x)$ (when M is even), can be found, and these are all collected below in

Theorem 2.

a) *If M is an odd integer, the fundamental polynomials $r_1(x)$ and $r_0(x)$ for the $(0; M)$ interpolation problem are given by*

$$r_1(x) = \frac{(-1)^{\frac{M+1}{2}}}{m} \left\{ \frac{\sin mx}{m^M} + 2\sum_{j=1}^{m-1} \frac{\sin[\,j(x - x_1)]}{[(m-j)^M - j^M]} \right\} (M \text{ odd}, m \text{ odd}, \varepsilon = 1) \tag{3.11}$$

and

$$r_0(x) = \frac{1}{m} \left\{ 1 + 2\sum_{j=1}^{m-1} \frac{(m-j)^M \cos jx}{[\,(m-j)^M - j^M]} \right\} (M \text{ odd}, m \text{ odd}, \varepsilon = 1). \tag{3.12}$$

b) *If M is an even positive integer, the fundamental polynomials $r_1(x)$ and $r_0(x)$ for the $(0; M)$ interpolation problem are given, for m an arbitrary positive integer, by*

$$r_1(x) = \frac{(-1)^{M/2}}{m} \left\{ \frac{1 - \cos mx}{m^M} + 2 \sum_{j=1}^{m-1} \frac{\cos[\,j(x - x_1)]}{[\,(m-j)^M + j^M]} \right\} (M \text{ even}, \varepsilon = 0), \tag{3.13}$$

and

$$r_0(x) = \frac{1}{m}\left\{1 + 2\sum_{j=1}^{m-1} \frac{(m-j)^M \cos jx}{[\,(m-j)^M + j^M]}\right\} (M \text{ even}, \ \varepsilon = 0), \tag{3.14}$$

for any positive integer m.

4. The Rate of Convergence of 2-Periodic Lacunary Trigonometric Interpolation. For any $f(x) \in C_{2\pi}$, consider its $(0; M)$-trigonometric interpolation with M *odd*, defined by $\sum_{j=0}^{m-1} f(x_{2j}) r_{2j}(x)$, the special case of (1.7) with $\beta_{2j+1} = 0 \ \ (0 \le j \le m-1)$. The following result gives an upper bound for $\|f(x) - \sum_{j=0}^{m-1} f(x_{2j}) r_{2j}(x)\|$ (where $\|g\| := \sup_x |g(x)|$), in terms of $E_s(f), (s \ge 0)$, the error of best uniform approximation to $f(x)$ by trigonometric polynomials of order at most s. As the method of Szabados [6] is easily adapted to this case, we omit the proof.

Theorem 3. *Let m and M be odd positive integers, let $\varepsilon = 1$ (cf. Theorem 1a), and let $r_{2j}(x)$ be the fundamental polynomials of (1.3). Then, for any $f(x) \in C_{2\pi}$,*

$$\|f(x) - \sum_{j=0}^{m-1} f(x_{2j}) r_{2j}(x)\| = O\left(m^{\frac{1-(-1)^M}{2}} E_{[m/4]}(f) + m^{-M} \sum_{k=0}^{m} (k+1)^{M-1} E_k(f)\right).$$

We remark that the O-symbol above denotes a constant depending only on M (and not m).

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