CONSTRUCTIVE THEORY OF FUNCTIONS'87

SOFIA, 1988

DESCARTES SYSTEMS IN HAAR SUBSPACES

Manfred Sommer and Hans Strauss

1. Introduction. Let M be a subset of $\mathbb{R}$ which contains at least n points ($n \geq 1$) and let $F(M) = \{f: M \to \mathbb{R}\}$. Moreover, let U denote an n-dimensional subspace of F(M). We say that U has a Descartes basis, if there exists a basis $\{u_1, \ldots, u_n\}$ of U such that for any integers $1 \leq i_1 < \ldots < i_m \leq n$ and any points $t_1 < \ldots < t_m$ in M,

$$D\begin{pmatrix} u_{i_1} & \ldots & u_{i_m} \\ t_1 & \ldots & t_m \end{pmatrix} = \det(u_{i_j}(t_k))_{j=1}^{m}{}_{k=1}^{m} \neq 0,$$

$1 \leq m \leq n$. The system $\{u_1, \ldots, u_n\}$ is called a Descartes system in U on M. Moreover, a system $\{u_1, \ldots, u_n\}$ in U is called a sign-regular Descartes system if for any integers $1 \leq i_1 < \ldots < i_m \leq n$ there exists an $\varepsilon_m \in \{-1, 1\}$ such that for any points $t_1 < \ldots < t_m$ in M,

$$\varepsilon_m D\begin{pmatrix} u_{i_1} & \ldots & u_{i_m} \\ t_1 & \ldots & t_m \end{pmatrix} > 0,$$

$1 \leq m \leq n$.

If M is an interval, then obviously every Descartes system is sign-regular and therefore, the terminology is consistent with that used by Karlin and Studden [1, p. 25].

It is wellknown that U is called a Haar subspace of F(M), if for any basis $\{u_1, \ldots, u_n\}$ of U and any points $t_1 < \ldots < t_n$ in M,

$$D\begin{pmatrix} u_1 & \ldots & u_n \\ t_1 & \ldots & t_n \end{pmatrix} \neq 0.$$

Hence, if $\{u_1, \ldots, u_n\}$ is a Descartes system in U on M, then every subsystem $\{u_{i_1}, \ldots, u_{i_m}\}$ of $\{u_1, \ldots, u_n\}$ spans a Haar subspace of U.

In the following we are interested in such Haar spaces which admit Descartes systems. We first give a sufficient condition ensuring the existence of Descartes systems. Under some weak additional hypotheses we are able to verify the more difficult converse result. In particular it follows that if M is a closed interval, then there exists a Descartes system in a subspace U of C(M) if and only if for every interval $\tilde{M} \supset M$ there exists a Haar subspace $\tilde{U}$ of $C(\tilde{M})$ such that $\tilde{U}|_M = U$. Moreover we show by an example that not every Haar space on M can be continuously extended to a Haar space defined on a set $\tilde{M} \supset M$. Finally we show that Descartes systems play an important role for existence of special bases in generalized spline spaces.

Independently and simultaneously Zalik and Zwick [4] studied the problem of existence of sign-regular Descartes systems in Haar spaces and obtained statements similar to our results, but using different methods.

2. The main results. We begin by giving a sufficient condition ensuring the existence of Descartes systems.

Theorem 1. Let U denote an n-dimensional Haar subspace of F(M). Assume that there exist distinct points $z_1, \ldots, z_n \in \mathbb{R} \setminus M$ and an n-dimensional Haar subspace $\tilde{U}$ of $F(\tilde{M})$ where $\tilde{M} = M \cup \{z_1, \ldots, z_n\}$ such that $\tilde{U}|_M = U$. Then there exists a Descartes system $\{u_1, \ldots, u_n\}$ in U on M.

The simple proof of this statement can be found in [2]. Under some weak additional hypotheses the more difficult converse of the above result is obtained.

Theorem 2. Let $\inf M \notin M$, $\sup M \notin M$, $a = \inf M > -\infty$ and assume that for any points $x, y \in M$ with $x < y$ there exists a point $z \in M$ with $x < z < y$. Set $\tilde{M} = M \cup \{a\}$. Assume that U is an n-dimensional subspace of $F(\tilde{M})$ which contains a sign-regular Descartes system on $\tilde{M}$. Then for every $d > 0$ U can be continuously extended to an n-dimensional Haar space U_d on $(a-d, a) \cup \tilde{M}$, i.e.

(i) $U_d|_{\tilde{M}} = U$;

(ii) every $u \in U_d$ is continuous on $(a-d, a]$.

The proof of this statement which can be found in [2] is long and complicated. Using many results on Haar spaces we show in that proof that if $U = \text{span}\{u_1, \dots, u_n\}$, then there exists a subspace $DU = \text{span}\{Du_1, \dots, Du_n\}$ of $F(\tilde{M})$ where $Du_1 \equiv 0$ and Du_i denotes a certain "generalized derivative" of u_i, $2 \leq i \leq n$. Moreover, it turns out that DU has the same properties as U. Then, since $\dim DU = n-1$, we proceed by induction on n. This implies that DU can be continuously extended to an (n-1)-dimensional Haar space DU_d on $(a-d,a) \cup \tilde{M}$. Finally, integrating DU_d we obtain the desired Haar space U_d.

For the most important case when M is an interval, the following equivalent statements are an immediate consequence of the above theorems.

Corollary 3. Let $M = [a,b)$, a real interval, and assume that U denotes an n-dimensional subspace of $C(M)$. Then the following conditions are equivalent:

(i) There exists a Descartes system in U on M;
(ii) For every $d > 0$ there exists an n-dimensional Haar subspace U_d of $C(\tilde{M})$ where $\tilde{M} = (a-d,b)$ such that $U_d|_M = U$.

Analogous statements hold if $M = [a,b]$ and $M = (a,b]$, respectively.

Obviously every Descartes system spans a Haar space. Now we show that the converse is not true in general. It was verified by Krein (see Zielke [5, Theorem 7.7]) that if $M = (a,b)$ and U is an n-dimensional Haar subspace of $C(M)$, then there exists a basis $\{u_1, \dots, u_n\}$ of U such that $\text{span}\{u_1, \dots, u_i\}$ is a Haar subspace of U, $1 \leq i \leq n$. (Such a basis is called a Markoff basis.) Zielke showed (see [5, Section 10]) that this statement fails, if $M = [a,b)$ or $M = [a,b]$. In particular this implies that there exist Haar spaces which do not contain Descartes systems. Therefore by Corollary 3, not every Haar space is extensible.

Example 4. Let $M = [-1,1]$ and let $U = \text{span}\{u_1, u_2, u_3\}$ where $u_1(x) = 1$, $u_2(x) = x(1-x)$ and $u_3(x) = (1-x^2)(1-x)$ for every $x \in [1,1]$. It was verified by Zielke (see [5, Section 10]) that U is a Haar space on $[-1,1]$. Moreover he showed that U does not contain a two-dimensional Haar subspace. This implies that U has no Descartes basis on $[-1,1]$. We have even shown in [2] that U has no Descartes basis on $(-1,1)$ and therefore, U cannot be extended to a Haar space on $(-1-d,1)$ or on $(-1,1+d)$, respectively, for any $d > 0$.

3. <u>An application to generalized spline spaces</u>. We show that Descartes systems play an important role for existence of special bases in generalized spline spaces. In [3] we introduced a class of such spaces as follows: Let $a = x_o < x_1 < \ldots < x_{k+1} = b$ denote a partition of $[a,b]$ $(k \geq 0)$ and let $V_i = \text{span}\,\{v_{i,1}, \ldots, v_{i,n_i}\}$ be a subspace of $C[x_i, x_{i+1}]$ with dimension $n_i \geq 0$ where $\{v_{i,1}, \ldots, v_{i,n_i}\}$ forms a Descartes system on $[x_i, x_{i+1}]$, $0 \leq i \leq k$. Suppose that p_i and r_i are nonnegative integers, $0 \leq i \leq k$ where $r_o = 0$ and $r_i \leq \min\,\{p_{i-1}, p_i\}$, $1 \leq i \leq k$. Moreover, let $r_{k+1} = 0$ and $p = \max\,\{p_i : 0 \leq i \leq k\}$ and suppose that $w_i \in C^{p-i}[a,b]$ are positive on $[a,b]$, $1 \leq i \leq p$.
If for some $i \in \{0, \ldots, k\}$, $p_i = 0$, we define $U_i = V_i$.
If for some $i \in \{0, \ldots, k\}$, $p_i > 0$, we define

$$U_i = \text{span}\,\{\{w_1(x) \int_{x_i}^{x} w_2(y_2) \int_{x_i}^{y_2} \ldots w_{p_i}(y_{p_i}) \int_{x_i}^{y_{p_i}} v(y_{p_i+1})$$

$$dy_{p_i+1} \ldots dy_2 : v \in V_i\} \cup \{\varphi_j\}_{j=1}^{p_i}\},$$

where $\varphi_1(x) = w_1(x)$,

$$\varphi_2(x) = w_1(x) \int_a^x w_2(y_2)\,dy_2,$$

$$\vdots$$

$$\varphi_{p_i}(x) = w_1(x) \int_a^x w_2(y_2) \ldots \int_a^{y_{p_i-1}} w_{p_i}(y_{p_i})\,dy_{p_i} \ldots dy_2.$$

Then the associated generalized spline space S is defined by

$$S = \{s : [a,b] \to \mathbb{R} : s\big|_{I_i} \in U_i,\ s_-^{(j)}(x_i) = s_+^{(j)}(x_i),\ 0 \leq j \leq r_i - 1,$$
$$1 \leq i \leq k\}$$

where $I_i = [x_i, x_{i+1})$, $0 \leq i \leq k-1$ and $I_k = [x_k, x_{k+1}]$.

<u>Theorem 5</u>. S has a basis $\{B_1, \ldots, B_n\}$ satisfying the following conditions:

(i) $\{B_1, \ldots, B_n\}$ forms a weak Descartes system, i.e. for any integers $1 \leq j_1 < \ldots < j_m \leq n$ and any points $a \leq t_1 < \ldots < t_m \leq b$,

$$D\begin{pmatrix} B_{j_1} & \dots & B_{j_m} \\ t_1 & \dots & t_m \end{pmatrix} \geq 0;$$

(ii)
$$D\begin{pmatrix} B_{j_1} & \dots & B_{j_m} \\ t_1 & \dots & t_m \end{pmatrix} > 0 \quad \text{if and only if}$$

$t_i \in T_{j_i}$, $1 \leq i \leq m$ where $T_j = \{x \in [a,b] : B_j(x) \neq 0\}$, $1 \leq j \leq n$;

(iii) T_j is a subinterval of $[a,b]$, $1 \leq j \leq n$, and

$\inf T_j \leq \inf T_{j+1}$, $\sup T_j \leq \sup T_{j+1}$, $1 \leq j \leq n-1$.

This statement which was proved in [3] shows the existence of basis functions of S with relatively small support. Moreover it was shown in [3] that these functions can be computed by a recursion relation. Therefore, they retain most of the features of the polynomial B-splines.

References

1. S. Karlin and W.J. Studden. Tchebycheff Systems: with Applications in Analysis and Statistics. Interscience, New York, 1966.
2. M. Sommer and H. Strauss. A characterization of Descartes systems in Haar subspaces. Preprint, 1986.
3. M. Sommer and H. Strauss. Weak Descartes systems in generalized spline spaces. Preprint, 1986.
4. R.A. Zalik and D. Zwick. On extending the domain of definition of Čebyšev and weak Čebyšev systems. Preprint, 1987.
5. R. Zielke, Discontinuous Čebyšev Systems. Lecture Notes in Mathematics 707. Springer, Berlin-Heidelberg-New York, 1979.

Mathematisch-Geographische Fakultät
Katholische Universität Eichstätt
8078 Eichstätt
Federal Republic of Germany

Institut für Angewandte Mathematik
Universität Erlangen-Nürnberg
8078 Erlangen
Federal Republic of Germany