CONSTRUCTIVE THEORY OF FUNCTIONS'87

SOFIA, 1988

EXTREMAL PROBLEMS ON SOME CLASSES OF DIFFERENTIABLE FUNCTIONS

Sun Yongsheng, Huan Daren, Fan Gensun

1. Preliminaries.

A basic class of differentiable functions which has long been extensively studied is the 2π-periodic differentiable function class represented by the convolution with the Bernoulli kernel $(2\pi)^{-1}\sum_{-\infty}^{\infty}{}' \frac{e^{i\nu x}}{(i\nu)^n}$. Its natural extension, the periodic convolution class with the generalized Bernoulli kernel will be discussed. Given a polynomial $P_n(\lambda)=\lambda^n + \sum_1^n a_j\lambda^{n-j}$ with real coefficients a_j, whose real factorization may be written as

$$P_n(\lambda)=\prod_{s=1}^{k}(\lambda^2-2\alpha_s\lambda+\alpha_s^2+\beta_s^2)\cdot\prod_{j=1}^{n-2k}(\lambda-\lambda_j).$$

Denote $\beta = \max \beta_s$ (when k=0 we set $\beta=0$). For simplicity we assume $P_n(ik)\neq 0$ for $k=\pm1,\pm2,\dots$, $i=\sqrt{-1}$.

Definition 1.1 The function

$$G_n(x)=(2\pi)^{-1}\sum_{-\infty}^{\infty}\frac{e^{i\nu x}}{P_n(i\nu)} \tag{1.1}$$

is said to be the generalized Bernoulli function relating to $P_n(D)$, where $D=\frac{d}{dx}$. It is first introduced by M.G.Krein [1] .

Definition 1.2 Say $f\in \mathfrak{W}_p^{(n)}(P_n(D))$, if

$$f(x)=C_0+\int_0^{2\pi}G_n(x-t)h(t)dt, \tag{1.2}$$

where $C_0=0$ when $P_n(0)\neq0$, C_0 is arbitrarily real when $P_n(0)=0$; $\|h\|_p\le 1$, $(1\le p\le +\infty)$, and $h\perp 1$.

Definition 1.3 The function $(\lambda>2\beta)$

$$\Phi_{n,\lambda}(x)=\frac{1}{\pi i}\sum_{-\infty}^{\infty}\frac{e^{i(2\nu+1)\lambda x}}{(2\nu+1)P_n((2\nu+1)\lambda i)} \tag{1.3}$$

is said to be the standard function relating to $P_n(D)$ in $\mathfrak{W}_\infty^{(n)}(P_n(D))$. Actually, it is the Euler spline determined by $P_n(D)$ with stepsize $\lambda^{-1}\pi$. It plays an important role in solving many optimal problems for the class $\mathfrak{W}_\infty^{(n)}(P_n(D))$ in optimal approximation. For example, denote Tm the totality of trigonometric polynomials of order$\le 2m+1$. The following result is well-known (cf. [1]).

Theorem 1.1 If $m > \Lambda \overset{df}{=} 4.3^{|H|}\beta$, then

$$E_m(\mathfrak{M}_p^{(n)}(P_n(D)))_p \overset{df}{=} \sup\{E_m(f)_p : f \in \mathfrak{M}_p^{(n)}\} =$$
$$= \sup\{\|f\|_p : f \in \mathfrak{M}_p^{(n)}, f \perp T_m\} = E_m(G_n)_1 = \|\Phi_{n,m}(\cdot)\|_c,$$

where $p \in \{1, +\infty\}$, and $E_m(f)_p \overset{df}{=} \min_{g \in T_{m-1}} \|f - g\|_p$.

2. Kolmogorov's Comparison Theorem and Landau-Kolmogorov type inqualities on $\mathbb{R}$ for $P_n(D)$.

In this section we given some results concerning the Kolmogorov's comparison theorem (cf. [3,8]) and L-K inequalities on $\mathbb{R}$ relating to $P_n(D)$. It is known that S.Karlin and others [2] [4] obtained a series of results in this respect.

Definition 2.1 If $P_r(\lambda)$ is a real factor of $P_n(\lambda)$ with degree r, then $P_r(D)$ is said to be a suboperator of $P_n(D)$. $\hat{P}_r(D)$ is the co-operator of $P_r(D)$ provided that $\hat{P}_r(\lambda).P_r(\lambda) \equiv P_n(\lambda)$.

Definition 2.2 Say $f \in L_p^{(n)}$, if $f \in C^{n-1}(\mathbb{R}) \cap L^p(\mathbb{R})$, $f^{(n-1)}$ is locally absolutely continuous on $\mathbb{R}$, and $\|P_n(D)f\|_p < +\infty$. Write

$$W_p^{(n)}(P_n(D)) = \{f \in L_p^{(n)} : \|P_n(D)f\|_p \le 1\}.$$

Theorem 2.1 (Sun, [5]) Given $n \geq 1, \lambda > \Lambda$. If $f \in W_\infty^{(n)}(P_n(D))$ satisfies

(1) $\|f\|_\infty \le \|\Phi_{n,\lambda}\|_\infty$.

(2) there are α, a, b $\in \mathbb{R}$ such that $f(\alpha) = \Phi_{n,\lambda}(a) = \Phi_{n,\lambda}(b)$, where a, b are chosen from two adjacent monotonic intervals of $\Phi_{n,\lambda}(x)$.

Then for any $\gamma \in \mathbb{C}$ it holds

$$|(D-\gamma)f(\alpha)| \le \max\{|(D-\gamma)\Phi_{n,\lambda}(a)|, |(D-\gamma)\Phi_{n,\lambda}(b)|\}. \tag{2.1}$$

Especially for a self-adjoint $P_n(D)$

$$|(D-\gamma)f(\alpha)| \le |(D-\gamma)\Phi_{n,\lambda}(a)|. \tag{2.2}$$

From Theorem 2.1 a series of important consequences may be derived.

Corollary 2.1 Under the assumption of Theorem 2.1 (without (2)) for any $\gamma \in \mathbb{C}$ it holds

$$\|(D-\gamma)f\|_\infty \le \|(D-\gamma)\Phi_{n,\lambda}\|_\infty. \tag{2.3}$$

This corollary contains a result of Ter Morsche and Scherer [6] .

Corollary 2.2 For any suboperator $P_r(D)$ of $P_n(D)$ with only real characterstic roots it holds

$$\|P_r(D)f\|_\infty \le \|P_r(D)\Phi_{n,\lambda}\|_\infty. \tag{2.4}$$

This corollary contains a result of Sharma and Tzimbalario [4] .

It is worth noting that corollary 2.2 may be further extended. Actually, we have

Theorem 2.2 (Sun, [7]) Under the assumption of Theorem 2.1 (without (2)) for any real suboperator $P_r(D)$ it holds (2.4).

Applying the technique of convolution transformation we have also established the Stein-type inquality. Besides this, by utilizing the rearrangement technique initiated by Korneichuk, [8] we have established the comparison theorem for the rearrangement of periodic functions from $L_{\infty}^{(n)}(P_n(D))$. As a consequence, the following Taikov-type inequatity is obtained.

Theorem 2.3 (Sun, [5]) Let $m>\Lambda$, $1\le p\le\infty$, $\frac{1}{p}+\frac{1}{p'}=1$. Then it holds

$$(1)\quad \sup\{\|f\|_p:\ f\in\mathfrak{M}_{\infty}^{(n)},\ f\perp T_{m-1}\}=\|\Phi_{n,m}\|_p. \tag{2.6}$$

$$(2)\quad E_m(\mathfrak{M}_p^{(n)})_1=\sup\{\|f\|_{p'}:\ f\in\mathfrak{M}_{\infty,*}^{(n)},\ f\perp T_{m-1}\}=\|\Phi_{n,m}\|_{p'} \tag{2.7}$$

where $\mathfrak{M}_{\infty,*}^{(n)}=\mathfrak{M}_{\infty}^{(n)}(P_n^*(D))$, $P_n^*(D)$ is the formally adjoint operator to $P_n(D)$.

Is it possible to extend the Landau-Kolmogorov inequality to some more general cases, for example, to a class of differentiable functions defined by a non-constant coefficients differential operator ? Chen Han-lin has pointed out in a recent paper [22] that this actually may be done for a differential operator $L_n=D_n\ldots\ldots D_1$, where $D_j f(t)=\alpha_{j-1}(t)D_t(\beta_{j-1}(t)f(t))$, $(j=1,\ldots,n+1)$, provided that some reasonable conditions are fulfilled by these $\alpha_j(t)$ and $\beta_j(t)$. To be precise, we assume that $\alpha_j(t)\in C^{n-j}(\mathbb{R})$, $\beta_j(t)\in C^{n+1-j}(\mathbb{R})$, $\alpha_j(t)>0$, $\beta_j(t)>0$ and $\alpha_j(t+h)=C_j\alpha_j(t)$, $\beta_j(t+h)=C_j^{-1}\beta_j(t)$, $C_j=\beta_j(0)\cdot\beta_j(h)^{-1}$, $h>0$. It is noted that when $\alpha_j(t)=\beta_j^{-1}(t)=e^{\lambda_j t}$ $(\lambda_j\in\mathbb{R})$ we have $L_j(t)=\prod_{s=0}^{j-1}(D-\lambda_s)$. Denote

$P_n=\{f: f\in C^{n-1}(\mathbb{R})$, $f^{(n)}$ bounded and piecewisely continous on $\mathbb{R}\}$

$$g^{\langle 0\rangle}=\|g\|_{\infty}.$$

$$g^{\langle j\rangle}=\sup_{\alpha s}\ \sup_{x}|L_j(\alpha_1,\cdots,\alpha_j)g(x)|,$$

where $L_j(\alpha_1,\cdots,\alpha_j)\ g(x)=D_jT(\alpha_j)\ldots D_1T(\alpha_1)g(x)$, $T(\alpha)$ is the translation operator $T(\alpha)g(x)=g(x-\alpha)$.

Theorem 2.4 (Chen, [22])

Let $g(x)\in P_n$, if $g^{\langle 0\rangle}\le\|L_0E_{n+1}(\cdot)\|_{\infty}$, $g^{\langle n\rangle}\le\|L_nE_{n+1}(\cdot)\|_{\infty}$, then it holds $g^{\langle j\rangle}\le\|L_jE_{n+1}(\cdot)\|_{\infty}$, $j=1,\ldots n-1$.

where $E_{n+1}(x)$ is the standard function relating to the operator Ln.

The behavier of $E_{n+1}(x)$ is something like the Euler spline function, but its characterization is too complicated to be quoted here.

3. Sharp Estimates of N-widths for some classes of differentiable functions.

In this section, in addition to $\mathfrak{M}_p^{(n)}(P_n(D))$, the following class of periodic functions will be considered.

Definition 3.1 Let $\omega(t)$ be a convex modulous of continuity. Say $f\in \mathfrak{M}^{(n)}H^{\omega}$, if $f=C_\sigma+G_n*h$, where h is 2π-periodic, $f\perp 1$, and $\omega(h,t)\le\omega(t)$. $\Psi_{g,m}(t)$ is an odd function with period $\frac{2\pi}{m}$ and satisfying $\Psi_{g,m}(t)=\Psi_{g,m}(\frac{\pi}{m}-t)$ $=\frac{1}{2}\omega(2t)$ on $[0,\frac{\pi}{2m}]$. $\Psi_{n,m}(t)=(G_n*\Psi_{g,m})(t)$.

Theorem 3.1 (Fan, [9,11]) for $m>\Lambda$.

(1) $d_{2m}(\mathfrak{M}^{(n)}H^{\omega}, C)\ge\|\Psi_{n,m}\|_C$. (3.1)

(2) $d_{2m}(\mathfrak{M}^{(n)}H^{\omega}, L)\le\|\Psi_{n,m}\|_1$. (for self-adjoint $P_n(D)$) (3.2)

(3)
$$d_{2m-1}(\mathfrak{M}_\infty^{(n)}, C)=d_{2m}(\mathfrak{M}_\infty^{(n)}, C)=d^{2m-1}(\mathfrak{M}_\infty^{(n)}, C)=d^{2m}(\mathfrak{M}_\infty^{(n)}, C)$$
$$=d'_{2m-1}(\mathfrak{M}_\infty^{(n)}, C)=d'_{2m}(\mathfrak{M}_\infty^{(n)}, C)=\|\Phi_{n,m}\|_C. \qquad (3.3)$$

In (3.3) T_{m-1} is optimal for d_N and d'_N, $N=2m-1$, $m>\Lambda$. When $N=2m$, there is a 2m-dimensional 2π-periodic $\mathcal{L}$-spline space determined by $P_n(D)$ to be optimal.

We notice that some results are obtained by Shevaltin [12] [13] . Our method of approach is different from therein. It is conjectured that for formally self-adjiont $P_n(D)$ in (3.1) and (3.2) should be equality.

Theorem 3.2 (Fan, [10]) For any $m>\Lambda$, $p\in\{1,2,+\infty\}$, or for any $m\ge 1$ $1\le p\le+\infty$ when $0\le\beta<\frac{1}{4}$ it holds

(1)
$$d_{2m}(\mathfrak{M}_\infty^{(n)}, L^p)=d'_{2m}(\mathfrak{M}_\infty^{(n)}, L^p)$$
$$=d^{2m}(\mathfrak{M}_\infty^{(n)}, L^p)=d^{2m-1}(\mathfrak{M}_\infty^{(n)}, L^p)=\|\Phi_{n,m}\|_p. \qquad (3.4)$$

There is a 2m-dimensional 2π-periodic $\mathcal{L}$-spline Space determined by $P_n(D)$ to be optimal for d_{2m} and d'_{2m} .

(2)
$$d_{2m-1}(\mathfrak{M}_p^{(n)}, L)=d_{2m}(\mathfrak{M}_p^{(n)}, L)=d'_{2m}(\mathfrak{M}_p^{(n)}, L)$$
$$=d^{2m}(\mathfrak{M}_p^{(n)}, L)=\|\Phi_{n,m}\|_{p'} \qquad (3.5)$$

Some optimal subspaces of d_{2m-1}, d_{2m} and d'_{2m} can also be indentified. The method of approach consists in applying a technique of convolution transform as well as a general form of Rolle's theorem related to a linear differential operator $P_n(D)$ with complex characteristic roots. It is conjectured that (3.4) and (3.5) should be true for any p, $1\le p\le+\infty$ and $m>\Lambda$.

In recent years very deep results have been obtained by A.Pinkus [24] and by Buslaev and Tikhomirov [25] respectively for the N-widths of Sobolev class W_p^r in L^q, $p\ge q$. We would like to show one result belonging to the circle of this problem. We turn to $P_n(D)=D^{\sigma}\prod_{j=1}^{l}(D^2-t_j^2)$, where $\sigma=0$ or 1, $l\ge 1$, $t_1,\dots,t_l\ge 0$, and $n=\sigma+2l$. Set
$\Omega_p^{(n)}=\{f: f^{(n-1)}$ abs. cont. on $[0,1]$, $f^{(2k+\sigma)}(0)=f^{(2k+\sigma)}(1)=0$, $k=0,\dots,l-1$, $\|P_n(D)f\|_p\le 1\}$.

Recently, the N-widths for $\Omega_p^{(n)}$ in L^p norm has been accurately calculated by Li Chun. His result is

Theorem 3.3 (Li, [14])

(1) When $\sigma=0$, for $N\geq 0$, $1<p<+\infty$ it holds

$$d_N(\Omega_p^{(n)}, \tilde{L}^p)=d^N(\Omega_p^{(n)}, \tilde{L}^p)=d'_N(\Omega_p^{(n)}, \tilde{L}^p)$$
$$=b_N(\Omega_p^{(n)}, \tilde{L}^p)=\| K_n * h_{N+1}\|_p. \tag{3.6}$$

$X_N^*=sp\{K_n(\cdot,\frac{1}{N+1}),\ldots,K_n(\cdot,\frac{N}{N+1})\}$ is optimal for d_N and d'_N. The optimal subspaces for d^N and b_N are also identified.

(2) When $\sigma=1$, it holds for $N\geq 0$, $1<p<+\infty$

$$d_0(\Omega_p^{(n)}, \tilde{L}^p)=d'_0(\Omega_p^{(n)}, \tilde{L}^p)=d^0(\Omega_p^{(n)}, \tilde{L}^p)$$
$$=b_0(\Omega_p^{(n)}, \tilde{L}^p)=+\infty. \tag{3.7}$$

for $N\geq 1$ the four width numbers all equal to $\| K_n * h_N\|_p$, and

$X_N^*=sp\{1, K_n(\cdot,\frac{1}{N}),\ldots,K_n(\cdot,\frac{N-1}{N})\}$ is optimal for d_N and d'_N.

$K_n(x,y)$ is the Green function for the differential operator $P_n(D)y=0$, $y^{(2k+\sigma)}(0)=y^{(2k+\sigma)}(1)=0$, $k=0,\ldots,l-1$, and $h_N(\cdot)$ is an odd function satisfying $h(\cdot+N^{-1})=-h_N(\cdot)$, $\|h_N\|_p=1$, and

$$\int_0^1 K_n(x,y)|(K_n*h_N)(x)|^{p-1}sgn((K_n*h_N)(x))\,dx$$
$$=\lambda_N^p|h_N(y)|^{p-1}sgn(h_N(y)). \qquad y\in[0,1]. \tag{3.8}$$

$\lambda_N>0$ is some number, and $sgn(h_N(x))=\pm sgn(\sin N\pi x)$. The existence and characterization of $h_N(x)$ and λ_N are proved by Li in his dissertation. We omit the details.

Li's results have obtained some further extension by Chen [23] to a more general case, where the class of functions under consideration are defined by the differential operator $L_n=D_n\ldots D_1$ introduced in the previous section with periodic or anti-periodic boundary conditions. In this case, the Green function, generally speaking, is no longer TP type. His method of approach consists in applying the theory of characteristic functions related to a linear differential operator with non-constant coefficients as well as a general form of Budan-Fourier theorem and some techniques. His paper will appear elsewhere.

To conclude this section we want to sketch briefly one result concerning the one-sided type N-widths for $\mathfrak{M}_1^{(n)}(P_n(D))$ in L-norm where $P_n(D)=D^\sigma\prod_{j=1}^{l}(D^2-t_j^2)$, $n=\sigma+2l$, $\sigma\geq 2$.

Theorem 3.4 (Sun and Huan, [15])

For $\sigma\geq 2$, $m\geq 1$, it holds

$$d_{2m-1}^+(\mathfrak{M}_1^{(n)}, L)=d_{2m}^+(\mathfrak{M}_1^{(n)}, L)=E_1(\Psi_m). \tag{3.9}$$

$$\Psi_m(t)=\frac{2\pi}{m}\sum G_n(t+\frac{2j\pi}{m})-\int_0^{2\pi}G_n(t)dt.$$

T_{m-1} is optimal. Besides this, there is a 2m-dimensional 2π-periodic $\mathcal{L}$-spline space determined by $P_n(D)$ to be optimal of d_{2m}^+. As a consequence, the exact value of $d_N^+[\widetilde{W}_1^{(m)}; L]$ is calculated.

For obtaining the above result we essentially applied the technique of Korneichuk, Doronin and Ligun [16], and relied upon the generalized Kolmogorov's comparison theorem (Theorem 2.1).

4. Convolution class with a kernel satisfying Chahkiev's condition.

Besides the Bernoulli kernel there are kernels of other character which play an important role in approximation theory. Of particular interest for optimal problems are the TP kernel (or CVD kernel) and the kernel satisfying Property B (cf. A. Pinkus [17]). Recently Chahkiev [18] [19] introduced the so-called RAq-type kernel.

Definition 4.1 (Chahkiev, [19]).

Let $\psi(t)$ be a 2π-periodic function in L_q. It is said to be satisfying Property RAq ($\psi \in$ RAq), provided that there exists a sequence of polynomials $P_r(t)$ of degree r such that each $P_r(t)$ has only real zeros and the corresponding Bernoulli functions $B_r(\cdot)$ satisfy

$$\|\psi(\cdot)-B_r(\cdot)\|_q \longrightarrow 0 \quad (r \rightarrow +\infty).$$

The following theorems characterize the function class RAq. Assume $\Lambda(t)$ to be a PF density (cf Hirschman and Widder [20]). Denote the Poisson sum of $\Lambda(t)$ by $\Phi(t)=\sum_{-\infty}^{\infty}\Lambda(t-2\nu\pi)$.

Theorem 4.1 (Fan, [21])

Let $\Lambda(t)$ be a PF density and $\Phi^{(-s)}(t)$ $(s\geq 0)$ be the s-th periodic integral of $\Phi(t)$ $(\Phi^{(0)}(t)=\Phi(t))$ such that $\Phi^{(-s)}(t)\perp 1$ for $s\geq 1$. Then $\Phi^{(-s)}(t) \in$ RAc, i.e. , there is a sequence of polynomials $P_r(t)$ such that each $P_r(t)$ has only real zeros and the corresponding generalized Bernoulli kernels $B_r(t)$ satisfy $\|\psi(\cdot)-B_r(\cdot)\|_c \rightarrow 0$ $(r \rightarrow +\infty)$.

Theorem 4.2 Assume $\psi(t)\in$ RAc and $\int_0^{2\pi}\psi(t)dt \neq 0$. Then $\psi(t) \in$ CVD (i.e. $\psi(t)$ possesses the cyclic-variation diminishing property).

Theorem 4.3 For a $\psi(t)\in$ RAc with $\int_0^{2\pi}\psi(t)dt=0$ it holds that $\psi(t)$ has Property B.

The significance of the condition RAq consists in that it makes possible to extend various exact results obtained for the generalized Bernoulli kernels to the kernels of more general type. Utilizing this method Chahkiev [19] solved some extremal problems concerning the best quadrature formula for a convolution class with a RAq kernel. Recently, Fan, applying this method, has succeeded in obtaining a series of new results concerning the exact estimation for the best one-sided L_1 approximation and the one-sided N-widths in L_1 norm on the convolution class of periodic functions with a RAq kernel. For example, he proved

Theorem 4.4 (Fan, [21]) Let $\psi(t)\in$ RAq. Suppose that there is a sequence of polynomials $P_r(t)=t^{\sigma}\prod_{j=1}^{l}(t^2-\lambda_{j,r}^2)$, where $r=\sigma+2l$, $\sigma\geq 2$, $\lambda_{j,r}\geq 0$

such that the corresponding generalized Bernoulli functions $B_r(t)$ satisfy $\|B_r(\cdot)-\Psi(\cdot)\|_c \to 0$ $(r \to +\infty)$. Then it holds

$$d^+_{2n-1}(K_1(\Psi), L) = d^+_{2n}(K_1(\Psi), L) = E_1(\Psi_n)_\infty,$$

where $K_1(\Psi)=\{f: f=C_0+\Psi*h,\ C_0=0$ and $\|h\|_1\le 1$ for even Ψ; C_0 arbitrary and $h\perp 1$, $\|h\|_1\le 1$ for odd $\Psi\}$, and

$$\Psi_n(t)=\frac{2\pi}{n}\sum_{j=1}^{n}\Psi\left(t+\frac{2j\pi}{n}\right)-\int_0^{2\pi}\Psi(t)dt.$$

T_{n-1} is optimal. Besides this, one can identify some 2n-dimensional 2π-periodic Ψ-spline space to be optimal for d^+_{2n} provided some rank condition is fulfilled by Ψ.

REFERENCE

1. М. Г. Крейн, ДАН СССР 18 (1938) 245-249.
2. S. Karlin, Oscillatory perfect splines and related extremal problems, in « Studies in Splines and Approximation Theory » Acad. Press (1976), 203-250.
3. Н. П. Корнейчук, Extremal Properties of Splines, in « Approximation Theory: Proc. of international conference held at Kaluga in 1975 » (In Russian), 1977.
4. Sharma, Tzimbalario, Illinois J. Math., 20 (1976) 443-455.
5. Sun Yongsheng, J. Beijing Normal Univ., (Natural Sci. Series) 4 (1984) 49-64.
6. H.G. Ter Morsche, Scherer, J. Approximation Theory, 43 (1985) 90-98.
7. Sun Yongsheng, « KEXUE TONGBAO » (Bulletin of Natural Sciences, in Chinese) 7, (1985).
8. Н. П. Корнейчук, « Экстремальные Задачи Теории Приближения » «НАУКА», М. 1976.
9. Fan Gensun, « KEXUE TONGBAO », 11, (1985) 806.
10. Fan Gensun, Chin. Ann. Math., 8A(1) (1987), 75-87.
11. Fan Gensun, North-eastern Math. J., 2(1) (1986) 17-32.
12. В. Т. Шевалдин, ДАН СССР 267 4(1982) 803-805.
13. В. Т. Шевалдин, Матем. Заметки, 33 (1983) 735-743.
14. Li Chun, Dissertation, Math. Institute of Chin. Acad. Sci., 1987.
15. Sun Yongsheng, Huan Daren, J. Appr. Theory and Appl. 1 (1) (1984), 19-35.
16. Н. П. Корнейчук, А. А. Лигун, В. Г. Доронин, « Аппроксимация с ограничениями » НАУКОВО ДУМКА Киев 1982.

17. A.Pinkus, J. D'Analyse Math., 35 (1979) 209-235.
18. М. А. Чахкиев, ДАН СССР 273 (1983) 60—65.
19. М. А. Чахкиев, Известия АН СССР сер. Матем. 48 (1984) 1078—1101.
20. I. Hirschman, D. Widder, « The Convolution Transform », Princeton Math. Series, 1955.
21. Fan Gensun, On PF density, Chahkiev's condition and some related optimal problems, (to appear).
22. Chen Han-lin, The Landau-Kolmogorov inequality for a differential operator with non-constant coefficients. (Preprint)
23. Chen Han-lin, On N-widths of periodic and anti-periodic functions, (Preprint)
24. A. Pinkus, Constr. Approx. (1985) 1, 15-62.
25. А. П. Буслаев, В. М. Тихомиров, ДАН СССР 283 (1985) 13—18.

Sun Yongsheng, Fan Gensun
Department of Mathematics
Beijing Normal University
People's Republic of China

Huan Daren
Department of Mathematics
Zhejiang University
People's Republic of China