CONSTRUCTIVE THEORY OF FUNCTIONS'87

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A THEOREM OF WHITNEY TYPE IN R^n

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1. Introduction. In [1] Bl. Sendov proved the following

Theorem 1. For any function f, bounded and integrable on $[0,1]$ and for each integer $n \geq 1$, there is a polynomial P of a degree at most n-1 such that

$$(1.1)\qquad |f(x) - P(x)| \leq 6\,\omega_n(f,1/(n+1)); \quad x \in [0,1] \text{ where}$$

$$\omega_n(f,\delta) = \sup\left\{ |\Delta_h^n f(x)| \;:\; |h| \leq \delta,\ x, x+nh \in [0,1] \right\};$$

$$\Delta_h^n f(x) := \sum_{i=0}^{n} (-1)^{n+1} \binom{n}{i} f(x+ih) \Big\}.$$

Ju. Brudnyi [2] proved an analogous theorem for approximation of functions of several variables by quasipolynomials. The purpose of this paper is to improve the Brudnyi's theorem in the sense of Theorem 1.

2. Definitions and denotations. Let x,y,... denote points in R^n and dx,dy,... denote Lebesque measure. A multi-index k is an n-tuple of nonnegative integers: $k = (k_1,\dots,k_n)$, $k_i \in \{0,1,\dots\}$, $i = 1,2,\dots,n$. We have the following definitions:

$$(2.1)\qquad |k| = k_1 + k_2 + \dots + k_n.$$

If k and l are two multi-indices:

$$(2.2)\qquad k \leq l \quad \text{iff} \quad k_i \leq l_i,\ i = 1,2,\dots,n.$$

$$(2.3)\qquad (k+l)_i = k_i + l_i,\ i = 1,\dots,n.$$

$$(2.4)\qquad (k-l)_i = \max(k_i - l_i, 0),\ i = 1,\dots,n.$$

$$(2.5)\qquad k! = (k_1!)(k_2!)\dots(k_n!),\ x^k = (x_1^{k_1})(x_2^{k_2})\dots(x_n^{k_n}).$$

$$(2.6)\qquad \binom{k}{l} = \binom{k_1}{l_1}\binom{k_2}{l_2}\cdots\binom{k_n}{l_n}$$

$$(2.7)\qquad \sum_{l \leq k} = \sum_{l_1=0}^{k_1}\sum_{l_2=0}^{k_2}\cdots\sum_{l_n=0}^{k_n}.$$

Let δ^i, $i = 1,2,\dots,n$, denote the multi-index which i-th component is 1 and the others are zero:

(2.8) $$\delta^i_j = \begin{cases} 1 & j=i, \\ 0 & j\neq i. \end{cases}$$

The multi-indexes α,β,γ we define as follows:

(2.9) $$0 \leq \alpha \leq 1, \quad \bar{\alpha} = 1 - \alpha .$$

(2.10) $$\int_0^y g\,dx = \int_0^{y_1} \dots \int_0^{y_n} g\,dx_1 \dots dx_n .$$

(2.11) $$\int_0^y g\,dx := \int_0^{y_{i_1}} \dots \int_0^{y_{i_{|\alpha|}}} g\,dx_{i_1} \dots dx_i \quad \text{where } \alpha_{i_j} = 1,$$

$j = 1,\dots,|\alpha|$.

By analogy with (2.11) we define $\binom{\alpha k}{l}$, $\prod_{j\leq\alpha k}$, $\sum_{j\leq\alpha k}$ and we denote:

(2.12) $$\hat{x}_j := (x_1,x_2,\dots,x_{j-1},x_{j+1},\dots,x_n).$$

Let m be any multi-index. Everywhere in this work m is fixed. We put:

(2.13) $$\begin{aligned} B^n &:= \left\{ x \in R^n:\ 0 \leq x_i \leq 1,\ i=1,\dots,n \right\}. \\ D &:= \left\{ x \in R^n:\ 0 \leq x_i \leq m_i+1,\ i=1,\dots,n \right\}, \\ D_\alpha &:= \left\{ x \in D:\ 0 \leq \alpha x \leq m,\ 1 \leq \bar{\alpha} x \leq m+1 \right\}. \end{aligned}$$

It is easy to see from (2.13) that

(2.14) $$D = \bigcup_{\alpha\leq 1} D_\alpha .$$

(2.15) $$D_i := \left\{ x \in D:\ 0 \leq x \leq m\,\bar{\delta}^i \right\}.$$

Let L(M) is the set of bounded and integrable in a Lebesque sense functions on the set M (where $M \subset R^s$ for some $s = 1,\dots,n$), equipped with the uniform norm $\|\cdot\|$. Here with $Q_m(D)$ we denote the set of all quasipolynomials such that the power of x_i does not exceed m_i ($i = 1,\dots,n$).

(2.16) $$Q_m(D) := \left\{ P(x) = \sum_{j=1}^{n} \sum_{k_i=0}^{m_i} f_{k_j}(\hat{x}_j)\, x_j^{k_j} \text{ where } f_{k_j}(x_j) \in L(D_j) \right\}$$

The best uniform approximation of the function $f(x)$ $L(D)$ by means of elements of $Q_m(D)$ is

(2.17) $$E(Q_m(D);f) := \inf\left\{ \|f-P\|:\ P \in Q_m(D) \right\}.$$

As a characteristic of $E(Q_m(D);f)$ we shall use the following modulus of smootness in R^n.

(2.18) $\omega_m(f;B^n) := \omega_m(f;\zeta) := \sup\{|\Delta_h^m f(x)|$: where $x, x+mh \in B^n$; $\zeta_i = 1/(m_i+1)$; $\Delta_h^m f(x) := \Delta_{h_1}^{m_1} \dots \Delta_{h_n}^{m_n} f(x)\}$.

We put $x = \mu + \sigma$ where $\sigma \in B^n$ and μ is a multi-index $\mu \leq m$.

$$(2.19) \qquad l_{m_i,q_i}(t) = \prod_{p=0,\neq q}^{m_i} (t-p)/(q_i-p) \qquad \text{for}$$

$q_i = 0,1,\dots,m_i$, are the basic Lagrange polynomials for the knots $0,1,\dots,m_i$. By analogy with (2.11) we define

$$(2.20) \qquad L_{m,q}^{\alpha}(\alpha t) := l_{m_{i_1},q_{i_1}}(t_{i_1}) \dots l_{m_{i_{|\alpha|}},q_{i_{|\alpha|}}}(t_{i_{|\alpha|}});$$

$j = 1,\dots,|\alpha|$, $i_j = 1$.

The following operator was introduced in [1] for R

$$(2.21) \qquad \varphi_r(f;t) := \varphi_r(f;p+s) := (-1)^{r+p}/\binom{r}{p} \int_0^1 \Delta_u^r f(x-pu)\,du$$

is defined for all functions $f \in L([0,r+1])$, where r,p are integers and $0 \leq s \leq 1$. Using (2.21) we define in $L(D)$ the operator $\varphi_m(f;x)$, which will play a very important role futher:

$$(2.22) \quad \varphi_m(f;x) := \varphi_m(f;\mu+\sigma) := (-1)^{|m|+|\mu|}/\binom{m}{\mu} \int_0^1 \Delta_u^m f(x-\mu u)\,du$$

for every $f(x) \in L(D)$. From (2.22) it is easy to see that:

$$(2.23) \qquad |\varphi_m(f;x)| \leq \omega_m(f;D)/\binom{m}{\mu} \qquad \text{for } \mu \leq x \leq \mu+1 \text{ and } \mu \leq m.$$

$$(2.24) \quad \varphi_{\alpha m}(f;x) := (-1)^{|\alpha m|+|\alpha\mu|}/\binom{\alpha m}{\mu} \int_0^{\alpha} \Delta_{\alpha u}^{\alpha m} f(x-\mu\alpha u)\,d\alpha u.$$

__3. Preliminaries.__ We shall use the following lemmas from [1]:

__Lemma 1.__

$$\max\left\{ \sum_{j=0}^{r} l_{r,j}(t) \;/\binom{r}{j} : \; 0 \leq t \leq 1 \right\} = 1.$$

__Lemma 2.__ Let

$$(3.1) \qquad \mu_{r,\nu} = \sum_{j=0}^{r} 1/\binom{r}{j} \max\left\{ |l_{r,j}(t)| \; 0 \leq t \leq \nu+1 \right\}$$

then $\mu_{r,\nu}$ $(1+\sigma_\nu+\sigma_{\nu+1})/\binom{r}{\nu}$ where $\nu = 0,1,\dots,[(r-1)/2]$

$\sigma_\nu = 1 + 1/2 + \dots + 1/\nu$; $\sigma_0 = 0$ and especially for $\nu = 0$

$$(3.2)\quad \mu_{r,o} = 1 + \sum_{j=0}^{r} 1/\binom{r}{j} \max\left\{ |l_{r,j}(t)| : 0 \le t \le 1 \right\} \le 2.$$

<u>Theorem 2.</u> Let $f \in L(D)$, then we have the following representation:

$$(3.3)\quad \int_0^{\mu+\sigma} f(u)du = \sum_{\alpha\le 1} \sum_{p=0}^{\alpha m} L^{\alpha}_{m,p}(\mu+\sigma) \int_0^{\alpha p} \sum_{q=0}^{\bar\alpha m} \int_0^{\bar\alpha\sigma} \varphi_{\bar\alpha m}(f(.);\bar\alpha(\mu+\sigma))$$
$$\cdot L^{\bar\alpha}_{m,q}(\mu+\sigma-u)d\bar\alpha u d\alpha u.$$

<u>Proof:</u>

It is evident from (2.24) that if $\beta \le \bar\alpha$ then $\alpha+\beta \le 1$

$$(3.4)\, \varphi_{\beta m}(\varphi_{\alpha m}(f;\alpha x)) = \varphi_{\alpha m}(\varphi_{\beta m}(f;\beta x)) = \varphi_{(\alpha+\beta)m}(f;(\alpha+\beta)x)$$

for every multi-index m.

We shall prove Theorem 2 by induction on n. for n = 1 from [1]

$$(3.5)\quad \int_0^{\mu+1} f(u)du = \sum_{p=0}^{r} l_{r,p}(\mu+1)\int_0^{p} f(u)du + \sum_{p=0}^{r} \int_0^{1} \varphi_r(f;p+u)l_{r,p}(\mu+1-u)du.$$

In order to complete the proof we apply (3.5) for the induction

$g(x) = \int_0^{\bar\alpha(\mu+1)} f(u)d\alpha u$ where $\alpha = 1 - \delta^n$. Theorem 2 follows from (3.4) and the induction assumption. □

If we differentiate (3.4) with respect to x_i for $i = 1,\dots,n$, the following basic representation of f(x) is obtained:

$$(3.5)\ f(x) - P^*(f;x) = \sum_{\beta\le 1} \sum_{q\le\beta m} \int_0^{\beta\sigma} \varphi_m(f(.);\beta(q+u)+\bar\beta(q+\sigma))$$
$$\cdot L'^{\beta}_{m,q}(\beta(\mu+\sigma-u))d\beta u, \text{ where}$$

$$P^*(f;x) = \sum_{0<\alpha\le 1} \sum_{p\le\alpha m} L'^{\alpha}_{m,p}(\alpha(\mu+\sigma)) \int_0^{\alpha p} \sum_{\beta\le\bar\alpha} \sum_{q\le\beta m} \int_0^{\beta\sigma} \varphi_{\bar\alpha m}(f(.);\beta(q+u)+\bar\beta(q+\sigma))$$
$$\cdot L'_{m,q}(\beta(\mu+\sigma-u))d\beta u d\alpha u.$$

<u>Theorem 3.</u> Let f L(D), then there is a quasipolynomial R(f;x) $Q_m(D)$ satisfying the following conditions:

(i) $\int_{\alpha j}^{\alpha(j+1)} (f(u) - R(f;u))d\alpha u = 0$ for any multi-indicex $\alpha \le 1$; $j \le m$

(ii) $\Delta_h^m R(f;x) = \sum_{0<\alpha\leq 1} (-1)^{|\alpha|} \int_0^{\alpha} \Delta^m_{\alpha+\bar{\alpha}h} f(\alpha u+\bar{\alpha}x)\,d\alpha u(\alpha h)^{\alpha m}$ where $0<h\leq 1$.

From (ii) we have

Corollary 1. $\omega_m(R;h) \leq \sum_{0<\alpha\leq 1}\omega_m(f;1) \leq (2^n-1)\omega_m(f;1)$.

Proof of Theorem 3. We denote

$F_\alpha(x) = \int_0^{\bar{\alpha}x} f(\alpha x+\bar{\alpha}u)\,d\bar{\alpha}u$ and set

(3.6) $R(f;x) = \sum_{0<\alpha\leq 1} (-1)^{|\alpha|+1} \sum_{p\leq\alpha m} F_{\bar{\alpha}}(\alpha p+\bar{\alpha}x)\, L^{\cdot\alpha}_{m+1,p}(\alpha x)$.

In order to prove (i) it is enough to show that (i) is true for $\alpha = \delta^i$ $(i = 1,\dots,n)$. We put $\beta = \alpha - \delta^i$ then $\bar{\alpha} = \bar{\beta} - \delta^i$.

$$\int_{\delta^i j}^{\delta^i (j+1)} \Big(f(x) - \sum_{0<\alpha\leq 1} (-1)^{|\alpha|+1} \sum_{p\leq\alpha m} F_{\bar{\alpha}}(\alpha p+\bar{\alpha}u) L^{\cdot\alpha}_{m+1,p}(\alpha u)\Big)\,d\delta^i u$$

$$= - \sum_{\substack{\alpha_i=0\\ 0<\alpha\leq 1}} (-1)^{|\alpha|+1} \sum_{p\leq\alpha m} \Big(F_{\bar{\alpha}-\delta^i}(\alpha p+\delta^i(j+1)+(\bar{\alpha}-\delta^i)u)$$

$$- F_{\bar{\alpha}-\delta^i}(\alpha p+\delta^i j+(\bar{\alpha}-\delta^i)u)\Big)L^{\cdot\alpha}_{m+1,p}(\alpha u)$$

$$+ \sum_{\substack{\beta_i=0\\ 0<\beta\leq 1}} (-1)^{|\beta|+1} \sum_{p\leq\beta m} \Big(F_{\bar{\beta}-\delta^i}(\beta p+\delta^i(j+1)+(\bar{\beta}-\delta^i)u)$$

$$- F_{\bar{\beta}-\delta^i}(\beta p+\delta^i j+(\bar{\beta}-\delta^i)u)\Big)L^{\beta}_{m+1,p}(\beta u) = 0.$$

Now we shall prove (ii).

$$\Delta_h^m R(f;x) = \Delta_h^m\Big(\sum_{0<\alpha\leq 1} (-1)^{|\alpha|+1} \sum_{p\leq\alpha m} F_{\bar{\alpha}}(\alpha p+\bar{\alpha}x) L^{\cdot\alpha}_{m+1,p}(\alpha x)\Big)$$

$$= \sum_{0<\alpha\leq 1} (-1)^{|\alpha|+1} \sum_{p\leq\alpha m} \int_0^{\alpha p} \Delta^{\bar{\alpha}m}_{\bar{\alpha}h} f(\alpha u+\bar{\alpha}x)\,d\alpha u \binom{\alpha(m+1)}{p} (-1)^{|\alpha(m+1)|+|\alpha p|}(\alpha h)^{\alpha m}$$

$$= \sum_{0<\alpha\leq 1} (-1)^{|\alpha|+1} \int_0^{\alpha} \sum_{p\leq\alpha m} (-1)^{|\alpha(m+1)|+|(\alpha p)|} \binom{\alpha m}{p} \Delta^{\bar{\alpha}m}_{\bar{\alpha}h} f(\alpha(u+p)\bar{\alpha}x)\,d\alpha x\, (\alpha h)^{\alpha m}$$

$$= \sum_{0<\alpha\leq 1} (-1)^{|\alpha|+1} \int_0^{\alpha} \Delta^m_{\bar{\alpha}h+\alpha} f(\alpha u+\bar{\alpha}x)\,d\alpha u(\alpha h)^{\alpha m}. \square$$

4. Main rezult. The purpose of this section is to prove the following

Theorem 4. If $f\in L(B^n)$, then there exist a quasipolynomial $P(x)\in Q_{m-1}(B^n)$ and a number $W(n)$ such that:

$$\|f(x) - P(x)\| \le W(n)\,\omega_m(f;B^n) \tag{4.1}$$

$$W(n) \le \sum_{i=0}^{n} \binom{n}{i} W(n-i) + 6^n \le 6^n\, n!, \text{ where } W(0) = 1. \tag{4.2}$$

Proof:

Let k be any multi-index. We consider the set $kB^n := \{kx : x\in B^n\}$. If in Theorem 4 we replase B^n by kB^n, it is clearly to see that smalest possible $W(n)$ does not depend on k. For simplicity we will prove Theorem 4 in the case $k = m$ (then $mB^n = D$).

We set $\bar{f}(x) = f(x) - R(f;x)$, $R(f;x)$ from Theorem 3, then from (3.5) and (2.13) we get

$$P^*_{D_\alpha}(g;x) = P^*_{D_\alpha}(f;x) = 0 \text{ where } g(x) := f(m_1+1-x_1,\dots,m_n+1-x_n). \tag{4.3}$$

From (3.5) we obtain

$$\bar{f}(x) = f(x) - R(f;x)$$
$$= \sum_{\beta\le 1}\ \sum_{q\le\beta m}\ \int_0^\beta \varphi_m(\bar{f}(.):\beta(q+u)+\bar\beta(q+\sigma))\, L^{\prime\beta}_{m,q}(\beta(\mu+\sigma-u))\,d\beta u.$$

$$\|f(x) - R(f:x)\| \le \sum_{\beta\le 1}\ \sum_{q\le\beta m} \omega_m(\bar{f};D)/\binom{\beta m}{\mu}\binom{\beta m}{q} \max_{\beta\mu\le t\le\beta(\mu+1)} L^{\beta}_{m,q}(\beta t).$$

Applying Lemma 1, Lemma 2, Corollary 1 we get

$$\|f(x) - R(f;x)\| \le \sum_{\beta\le 1} \omega_m(\bar{f}:D)/\binom{m}{\mu} \prod_{\beta_i\ne 0} (1+\sigma_i+\sigma_{i+1})$$
$$\sum_{\beta\le 1} \omega_m(\bar{f};D) \prod_{\beta_i\ne 0} 2 \le \sum_{\beta\le 1} \omega_m(\bar{f};D)\, 2^{|\beta|} \le \omega_m(\bar{f};D) \sum_{i=0}^{n}\binom{n}{i} 2^i.$$

$$\|f(x) - R(f;x)\| \le 6^n\omega_m(f;D). \tag{4.4}$$

$$R(f;x) = P(f;x) + \int_0^\alpha \Delta^{\alpha m}_{\alpha} f(\alpha u+\bar\alpha x)\,d\alpha u \prod_{j\le\alpha(m+1)} \alpha(x-j)/(\alpha m)!,$$ where

$P(f;x)\in Q_{m-1}(D)$.

$$\|f(x) - P(f;x)\| \le \sum_{0<\alpha\le 1} W(|\bar\alpha|)\,\omega_m(f;D) = \sum_{i=1}^{n}\binom{n}{i} W(n-i)\,\omega_m(f;D), \tag{4.5}$$

where $W(0) = 1$.

From (3.4) and (3.5) we obtain

$$W(n) \leq \sum_{i=1}^{n} \binom{n}{i} W(n-i) + 6^n \leq 6^n n! \;.\Box$$

In [3] it is proved that $W(2) \leq 49$.

References

1. Bl. Sendov, On the theorem and constants of H. Whitney, Constr. Approx.,3,1987,1-11.

2. Ю. А. Брудный, Приближение функций n переменных квазимногочленами, Изв. АН ССР сер. матем., 34,1970,564-583.

3. Bl. Sendov, M. Takev, A theorem of Whitney's type in R^2, (to appear).

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