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BEST BILINEAR APPROXIMATION AND CONNECTED QUESTIONS

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1. Let $x=(x_1,\dots,x_d)$, $y=(y_1,\dots,y_d)$ and $\pi_d=\prod_{j=1}^{d}[0,2\pi)$. For $\mathbb{P}=(\underbrace{p_1,\dots,p_1}_{d},\underbrace{p_2,\dots,p_2}_{d})$ $L_{\mathbb{P}}(\pi_{2d})$ will denote the set of functions $f(x,y)$, $x,y\in\pi_d$, of period 2π in each variable such that the mixed norm

$$\|f(x,y)\|_{\mathbb{P}}=\|\|f(\cdot,y)\|_{p_1}\|_{p_2}$$

is finite

In this note we shall study the best bilinear approximations in $L_{\mathbb{P}}(\pi_{2d})$ of functions of the Sobolev classes $SW^{\mathbb{R}}_{q,\alpha}$ and the Nikol'skii classes $NH^{\mathbb{R}}_{q}$, $\mathbb{R}=(\mathbb{R}^1,\mathbb{R}^2)$, $\mathbb{R}^i=(R^i_1,\dots,R^i_d)$,

$$i=1,2\ ,\quad q=(q^1,q^2)\ ,\quad q^i_j=q_i\ ,\ j=1,\dots,d\ ,\ i=1,2\ .$$

For a function $f(x,y)\in L_{\mathbb{P}}(\pi_{2d})$ we define the value of the best bilinear approximation of order M as follows:

$$\tau_M(f)_{\mathbb{P}}=\inf_{\substack{u_i(x),\,v_i(y)\\ i=1,\dots,M}}\|f(x,y)-\sum_{i=1}^{M}u_i(x)v_i(y)\|_{\mathbb{P}}\ ,$$

where itis assumed that $u_i(x)\in L_{p_1}(\pi_d)$, $v_i(y)\in L_{p_2}(\pi_d)$, $i=1,\dots,M$, and for a class of functions $F\subset L_{\mathbb{P}}(\pi_{2d})$

$$\tau_M(F)_{\mathbb{P}}=\sup_{f\in F}\tau_M(f)_{\mathbb{P}}\ .$$

We now give the definitions of the Sobolev and Nikol'skii classes. Let $F_R(t,\alpha)$, $t\in[0,2\pi)$ denote the one-dimensional Bernoulli kernel, i.e.

$$F_R(t,\alpha) = 2\sum_{k=1}^{\infty} k^{-R}\cos(kt-\alpha\pi/2)$$

The Sobolev class $SW_{q,\alpha}^{\mathbb{R}}B$ of functions of n variables consists of the functions $f(z)$, $z=(z_1,\dots,z_n)\in\pi_n$, which admit for each $1\leqslant j\leqslant n$ an integral representation

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi}\varphi_j(z^j, z_j-t)F_{R_j}(t,\alpha_j)\,dt\ ,$$

where $z^j=(z_1,\dots,z_{j-1},z_{j+1},\dots,z_n)$ and $\|\varphi_j\|_q\leqslant B$, $j=1,\dots,n$.

The Nikol'skii class $NH_q^{\mathbb{R}}B$ is the set of functions $f(z)\in L_q(\pi_n)$ such that for $l_j=[R_j]+1$

$$\|\Delta_\delta^{l_j}f(z)\|_q \leqslant B\,|\delta|^{R_j}\ , \quad j=1,\dots,n,$$

where $\Delta_\delta^{l_j}$ is the l_j -th difference with step δ in variable z_j.

In addition, for convenience we set

$$\int_0^{2\pi} f(z)\,dz_j = 0\ , \quad j=1,\dots,n$$

In the definitions of the Sobolev and Nikol'skii classes there are $R_j>0$, $j=1,\dots,n$. We shall denote by $NH_q^{\mathbb{R},0}B$ the set of functions $f(x,y)\in L_q(\pi_{2d})$, $x,y\in\pi_d$, such that for each $y\in\pi_d$, $f(\cdot,y)\in NH_{q_1}^{\mathbb{R}}B(y)$ and $\|B(y)\|_{q_2}\leqslant B$.

For convenience we denote by $F_q^{\mathbb{R}}$ one of the classes $SW_{q,\alpha}^{\mathbb{R}}B$ or $NH_q^{\mathbb{R}}B$ in the case $n=2d$.

2. This section continues (see the author's papers [1-2] where references to other papers on the same circle of questions are also given) the investigation of problems on the best bilinear approximations. Let for $\mathbb{R}=(R_1,\dots,R_n)$, $R_j>0$, $j=1,\dots,n$, $g(\mathbb{R})=\left(\sum_{j=1}^{n}R_j^{-1}\right)^{-1}$ and $F_q^{\mathbb{R}}$ denote one of the classes of functions $f(x,y)$ of $2d$ variables $SW_{q,\alpha}^{\mathbb{R}}B$ or $NH_q^{\mathbb{R}}B$. We have obtained the following assertions.

Theorem 2.1. Let $\mathbb{R} = (\mathbb{R}^1, \mathbb{R}^2)$, $g(\mathbb{R}^1) \leq g(\mathbb{R}^2)$, $\mathbb{q} \leq \mathbb{P}$, $1 \leq p_1 \leq 2$, $g(\mathbb{R}^1) > \beta_1$, $g(\mathbb{R}^2) > \beta_2 (1 - \beta_1 / g(\mathbb{R}^1))^{-1}$, where $\beta_i = \frac{1}{q_i} - \frac{1}{p_i}$, $i = 1,2$. Then

$$\tau_M (F_{\mathbb{q}}^{\mathbb{R}})_{\mathbb{P}} \asymp M^{-g(\mathbb{R}^2) + \frac{g(\mathbb{R}^2)}{g(\mathbb{R}^1)} \beta_1}$$

Theorem 2.2. Let $\mathbb{R} = (\mathbb{R}^1, \mathbb{R}^2)$, $g(\mathbb{R}^1) > g(\mathbb{R}^2)$, $\mathbb{q} \leq \mathbb{P}$, $1 \leq q_1 \leq 2$, $1 \leq q_2 \leq \infty$, $g(\mathbb{R}^2) > \frac{1}{q_2}$, $g(\mathbb{R}^1) > \frac{1}{q_1}\left(1 - \frac{1}{q_2 g(\mathbb{R}^2)}\right)^{-1}$.

Then

$$\tau_M (F_{\mathbb{q}}^{\mathbb{R}})_{p_1, \infty} \asymp M^{-g(\mathbb{R}^1) + \left(\frac{g(\mathbb{R}^1)}{g(\mathbb{R}^2)} - 1\right)\frac{1}{q_2} + \frac{1}{q_1} - \max\left(\frac{1}{2}, \frac{1}{p_1}\right)}, \quad 1 \leq p_1 \leq \infty$$

Theorem 2.3. Let $\mathbb{R} = (\mathbb{R}^1, \mathbb{R}^2)$, $g(\mathbb{R}^1) > g(\mathbb{R}^2)$, $\mathbb{q} \leq \mathbb{2}$, $g(\mathbb{R}^2) > \frac{1}{q_2} - \frac{1}{2}$, $g(\mathbb{R}^1) > \left(\frac{1}{q_1} - \frac{1}{2}\right)\left(1 - \left(\frac{1}{q_2} - \frac{1}{2}\right) g(\mathbb{R}^2)^{-1}\right)^{-1}$.

Then

$$\tau_M (F_{\mathbb{q}}^{\mathbb{R}})_{\mathbb{2}} \asymp M^{-g(\mathbb{R}^1) + \frac{g(\mathbb{R}^1)}{g(\mathbb{R}^2)}\left(\frac{1}{q_2} - \frac{1}{2}\right) + \frac{1}{q_1} - \frac{1}{q_2}}.$$

Theorem 2.4. Let $\mathbb{q}, \mathbb{P}$ be such that $1 \leq q_1 \leq p_1 \leq \infty$, $1 \leq q_2 = p_2 \leq \infty$. Then for $g(\mathbb{R}) > r(q_1, p_1)$ we have

$$\tau_M (NH_{\mathbb{q}}^{\mathbb{R}, \mathbb{O}} B)_{\mathbb{P}} \asymp M^{-g(\mathbb{R}) + \left(\frac{1}{q_1} - \max\left(\frac{1}{2}, \frac{1}{p_1}\right)\right)_+},$$

where

$$r(q, p) = \begin{cases} \left(\frac{1}{q} - \frac{1}{p}\right)_+ & \text{for } 1 \leq q \leq p \leq 2 \,;\; 1 \leq p \leq q \leq \infty, \\ \max\left(\frac{1}{2}, \frac{1}{q}\right) & \text{for } 1 \leq q \leq p \leq \infty, \quad p \geq 2 \end{cases}$$

3. Let $f(x,y) \in L_2(\pi_{2d})$ and $\mathcal{J}_f$ denote the integral operator defined by

$$(\mathcal{J}_f \varphi)(x) = (2\pi)^{-d} \int_{\pi_d} f(x,y) \varphi(y) \, dy \quad \text{for} \quad \varphi \in L_2(\pi_d).$$

Let $s_n(\mathcal{I}_f)$ be the singular values of operator $\mathcal{I}_f$ (i.e. $s_n(\mathcal{I}_f) = \lambda_n(\mathcal{I}_f \mathcal{I}_f^*)^{\frac{1}{2}}$, $\lambda_n(G)$ - eigenvalues of operator G arranged in decreasing order.

We denote for a class of functions $F \subset L_2(\pi_{2d})$

$$s_n(F) = \sup_{f \in F} s_n(\mathcal{I}_f).$$

We have obtained the following assertions.

Theorem 3.1. Let $1 \leqslant q_1 \leqslant \infty$, $q_2 = 2$ and $g(\mathbb{R}) > r(q_1, 2)$. Then

$$s_n(NH_q^{\mathbb{R},0}B) \asymp n^{-g(\mathbb{R}) + \max\left(\frac{1}{q_1}, \frac{1}{2}\right) - 1}$$

There is a known result of F.Smithies [3] for the case $d = 1$ (other results of I.Fredholm, H.Weyl, E.Hille and I.D.Tamarkin are cited in [3]):

$$s_n(NH_{(q,2)}^{R,0}B) \ll n^{-R + \frac{1}{q} - 1}, \quad 1 < q \leqslant 2, \quad R > \frac{1}{q} - \frac{1}{2}.$$

Theorem 3.2. Let $\mathbb{R} = (\mathbb{R}^1, \mathbb{R}^2)$, $g(\mathbb{R}^1) \leqslant g(\mathbb{R}^2)$, $q \leqslant 2$,

$$g(\mathbb{R}^1) > \frac{1}{q_1} - \frac{1}{2}, \quad g(\mathbb{R}^2) > \left(\frac{1}{q_2} - \frac{1}{2}\right)\left(1 - \left(\frac{1}{q_1} - \frac{1}{2}\right) g(\mathbb{R}^1)^{-1}\right)^{-1}.$$

Then

$$s_n(F_q^{\mathbb{R}}) \asymp n^{-g(\mathbb{R}^2) + \frac{g(\mathbb{R}^2)}{g(\mathbb{R}^1)}\left(\frac{1}{q_1} - \frac{1}{2}\right) - \frac{1}{2}}$$

Theorem 3.3. Let $\mathbb{R} = (\mathbb{R}^1, \mathbb{R}^2)$, $g(\mathbb{R}^1) > g(\mathbb{R}^2)$, $q \leqslant 2$,

$$g(\mathbb{R}^2) > \frac{1}{q_2} - \frac{1}{2}, \quad g(\mathbb{R}^1) > \left(\frac{1}{q_1} - \frac{1}{2}\right)\left(1 - \left(\frac{1}{q_2} - \frac{1}{2}\right) g(\mathbb{R}^2)^{-1}\right)^{-1}$$

Then

$$s_n(F_q^{\mathbb{R}}) \asymp n^{-g(\mathbb{R}^1) + \frac{g(\mathbb{R}^1)}{g(\mathbb{R}^2)}\left(\frac{1}{q_2} - \frac{1}{2}\right) + \frac{1}{q_1} - \frac{1}{q_2} - \frac{1}{2}}$$

4. This section continues the investigation (started in [2]) of the problems of the estimation of Kolmogorov width of classes of functions wich have integral representation with the kernel belonging to a certain class.

Let function $h(x,y) \in L_{p,q'}(\pi_{2d})$ be given. Consider the set W_q^h of functions $\psi(x)$ representable in the form

$$\psi(x) = (2\pi)^{-d} \int_{\pi_d} h(x,y)\varphi(y)\,dy \;, \quad \|\varphi\|_q \leq 1 \;, \quad \frac{1}{q} + \frac{1}{q'} = 1 .$$

and consider Kolmogorov width of this class:

$$d_M(W_q^h)_p = \inf_{\{u_i(x)\}_{i=1}^M} \; \sup_{\psi \in W_q^h} \; \inf_{v_i} \Big\| \psi(x) - \sum_{i=1}^M v_i u_i(x) \Big\|_p .$$

The following propositions hold.

Theorem 4.1. Let $\mathbb{R} = (\mathbb{R}^1, \mathbb{R}^2)$, $g(\mathbb{R}^1) \leq g(\mathbb{R}^2)$, $1 \leq q_1 \leq p \leq 2$,

$$g(\mathbb{R}^1) > \frac{1}{q_1} - \frac{1}{p} \;, \quad g(\mathbb{R}^2) > \frac{1}{q_2}\left(1 - \left(\frac{1}{q_1} - \frac{1}{p}\right) g(\mathbb{R}^1)^{-1}\right)^{-1} .$$

Then

$$\sup_{h \in F_q^{\mathbb{R}}} d_M(W_1^h)_p \asymp M^{-g(\mathbb{R}^2) + \frac{g(\mathbb{R}^2)}{g(\mathbb{R}^1)}\left(\frac{1}{q_1} - \frac{1}{p}\right)}$$

Theorem 4.2. Let $\mathbb{R} = (\mathbb{R}^1, \mathbb{R}^2)$, $g(\mathbb{R}^1) > g(\mathbb{R}^2)$, $1 \leq q_1 \leq \min(p, 2)$,

$$1 \leq p \leq \infty \;; \quad g(\mathbb{R}^2) > \frac{1}{q_2} \;, \quad g(\mathbb{R}^1) > \frac{1}{q_1}\left(1 - \frac{1}{q_2\, g(\mathbb{R}^2)}\right)^{-1} .$$

Then

$$\sup_{h \in F_q^{\mathbb{R}}} d_M(W_1^h)_p \asymp M^{-g(\mathbb{R}^1) + \left(\frac{g(\mathbb{R}^1)}{g(\mathbb{R}^2)} - 1\right)\frac{1}{q_2} + \frac{1}{q_1} - \max\left(\frac{1}{2}, \frac{1}{p}\right)} .$$

Theorem 4.3. Let $\mathbb{R} = (\mathbb{R}^1, \mathbb{R}^2)$, $g(\mathbb{R}^i) > 1$, $i = 1, 2$, $1 \le p \le q \le \infty$.

Then

$$\sup_{h \in F_1^{\mathbb{R}}} d_M(W_q^h)_p \asymp \begin{cases} M^{-g(\mathbb{R}^2) + \left(1 - \frac{1}{p}\right)\left(\frac{g(\mathbb{R}^2)}{g(\mathbb{R}^1)} - 1\right)}, & g(\mathbb{R}^1) \le g(\mathbb{R}^2), \\ M^{-g(\mathbb{R}^1) + \frac{1}{q}\left(\frac{g(\mathbb{R}^1)}{g(\mathbb{R}^2)} - 1\right)}, & g(\mathbb{R}^1) \ge g(\mathbb{R}^2). \end{cases}$$

Theorem 4.4. Let $\mathbb{R} = (\mathbb{R}^1, \mathbb{R}^2)$, $g(\mathbb{R}^i) > 1 + \tau(q, p)$, $1 \le q \le p \le \infty$, $i = 1, 2$.

Then

$$\sup_{h \in F_1^{\mathbb{R}}} d_M(W_q^h)_p \asymp \begin{cases} M^{-g(\mathbb{R}^2) + \left(1 - \frac{1}{p}\right)\left(\frac{g(\mathbb{R}^2)}{g(\mathbb{R}^1)} - 1\right) + \left(\frac{1}{q} - \frac{1}{2}\right)_+ - \left(\frac{1}{p} - \frac{1}{2}\right)_+}, & g(\mathbb{R}^1) \le g(\mathbb{R}^2) \text{ and } p \le 2 \text{ or } q \ge 2, \\ M^{-g(\mathbb{R}^1) + \frac{1}{q}\left(\frac{g(\mathbb{R}^1)}{g(\mathbb{R}^2)} - 1\right) + \left(\frac{1}{q} - \frac{1}{2}\right)_+ - \left(\frac{1}{p} - \frac{1}{2}\right)_+}, & g(\mathbb{R}^1) \ge g(\mathbb{R}^2). \end{cases}$$

We call attention to the recent paper[4], where some estimates of Kolmogorov width are obtained.

Bibliography

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