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ON THE PRECISION OF THE POLYGONAL INTERPOLATION IN H^s_ω ($s \geq 2$)

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1. Introduction. Let Ω be a given domain in R^s ($s \geq 2$), the knot sequence $X = \{ x_1, x_2, \ldots, x_m \} \subset \Omega$ and T_i ($i=1,2,\ldots,n$) is a ($s+1$) - simplex which is formed from ($s+1$) points of X such that $\Omega = \bigcup_{i=1}^{n} T_i$, int ($T_i \cap T_j$) = $\emptyset$ ($i \neq j$). Let us denote by T (Ω) the set of all such divisions $\{ T_i \}$ of Ω .

We shall write ($\Sigma \subset T(\Omega)$) :

$$d_\Sigma = \max \{ d_T : T \in \Sigma \} ,$$

where $K(O,R)$ is the circumsphere of T with center O and radius R)

$$(1) \quad d_T = \begin{cases} 2R, & O \in T, \\ \mathrm{diam}(T), & O \notin T. \end{cases}$$

For any bounded function f on Ω and $\Sigma \in T(\Omega)$ we introduce the linear interpolant $L(f, \Sigma ; x)$, $x \in \Omega$ as the complemented graphic [3] of $\tilde{L}(f, \Sigma ; x)$:

$$\tilde{L}(f, \Sigma ; x) = \alpha^i_1 x_1 + \alpha^i_2 x_2 + \ldots + \alpha^i_s x_s + \alpha^i_{s+1} \quad (T \in \Sigma),$$

$$\tilde{L}(f, \Sigma ; x_j) = f(x_j), \quad x_j \in X.$$

For any modulus of continuity $\omega(\delta)$, $0 \le \delta \le \operatorname{diam}(\Omega)$, $(\omega(0) = 0;\ \omega(\delta_1) \le \omega(\delta_2)$ if $\delta_1 \le \delta_2$; $\omega(\delta_1+\delta_2) \le \omega(\delta_1) + \omega(\delta_2))$ let us denote by $H^s_\omega(\Omega)$ the class of all functions f defined on Ω, for which (ρ -Eucleadean distance in R^s):

$$\omega(f,\delta) = \sup\{\,|f(X)-F(Y)| \ :\ \rho(X,Y) \le \delta\ ;\ X,Y \in \Omega\,\} \le \omega(\delta).$$

We shall briefly write (when Σ is fixed) d (instead d_Σ), $L(f;X)$ (instead $L(f,\Sigma;X)$), etc.

Let

$$C_{k,s}(\omega,\Sigma) = \sup\left\{\|f(.)-L(.)\|_{C(\Omega)}/\omega(f;\tfrac{d}{k}) : f \in H^s_\omega(\Omega)\right\},$$

$$C_{k,s} = \sup\{\,C_{k,s}(\omega,\Sigma) : \omega(\delta) \not\equiv 0\ ;\ \Sigma\,\}, \quad s=1,2,\ldots,$$

$k=1,2,\ldots$,

where as usual $\|f\|_{C(\Omega)} = \sup\{\,|f(X)| : X \in \Omega\,\}$.

In the case ω - a convex modulus of continuity we shall write $C^*_{k,s}(\omega,\Sigma)$ and $C^*_{k,s}$.

The following results are known:

<u>Theorem A</u>. For s=1,2 we have(k=2,s=1 [1] ,k=2,s=2 [4,5],s=1,2, k=1,2,.... [6]):

(2) $$C^*_{k,1} = \frac{k}{2},$$

(3) $$C^*_{2k,2}(\omega,\Sigma) = k.$$

<u>Theorem B [2]</u> . For s =1,2 and k=1,2,...

(4) $$C_{k,1}(\omega,\Sigma) \le \frac{k+1}{2},$$

(5) $$C_{k,1} = \frac{k+1}{2}.$$

The inequality (4) is strict for even k and (s=2,k=1,2,... [6])

(6) $$C_{2k,2}(\omega,\Sigma) < k+1,$$

(7) $$C_{2k,2} = k+1.$$

Theorem C. For any $\alpha \in [k,k+1)$, $k=1,2,\ldots$, there is a moddulus of continuity $\omega(\delta) \neq 0$ and a partition $\Sigma \in T(\Omega)$ such that [6]

(8) $c_{k,1}(\omega, \Sigma) = \frac{\alpha}{2}$,

(9) $c_{2k,2}(\omega, \Sigma) = \alpha$.

If in addition k is odd, then Theorem C holds for $\alpha = k+1$, as well. The case $k=2$ ($\alpha \in [1,\frac{3}{2})$) is established in [2] .

2. Main Results.

Theorem 1. For $s=2,3,\ldots,k=1,2,\ldots$ we have

(10) $c^{*}_{2k,s}(\omega, \Sigma) = k$.

Theorem 2. For $s=2,3,\ldots,k=1,2,\ldots$ we have

(11) $c_{2k,s}(\omega, \Sigma) < k+1$,

(12) $c_{2k,s} = k+1$.

Theorem 3. For any $\alpha \in [k,k+1)$, $s=2,3,\ldots$, $k=1,2,\ldots$ there is a modulus of continuity $\omega(\delta)$, domain Ω and a partition $\Sigma \in T(\Omega)$ such that

(13) $c_{2k,s}(\omega, \Sigma) = \alpha$.

3. The Proofs of Theorem 1 and Theorem 2 follow the scheme in [6]. For the proof of Theorem 3 let us fix $R > 0$ and k. Let $a \in (0,\frac{R}{k})$ and

$$\omega(x) = \begin{cases} \frac{x}{a} & , x \in [0,a), \\ 1 & , x \in [a, \frac{R}{k}], \\ \left[\frac{xk}{R}\right] + \omega\left(x - \frac{R}{k}\left[\frac{xk}{R}\right]\right), & x > \frac{R}{k} , \end{cases}$$

where $[x]$ is the integer part of x. Now, let $T=A_1A_2\ldots A_{s+1}$ be a $s+1$ - simplex for which $A_2A_3\ldots A_{s+1}$ is a regular s-simplex .

with centtrum $O(0,0,\dots,0)$ of the circumsphere and radius R, and $OA_1 \perp A_2A_3\dots A_{s+1}$, $OA_1 = R$.

So, $A_i(0,a_2^{(i)},a_3^{(i)},\dots,a_{s+1}^{(i)})$, $i=1,2,\dots,s+1$, $A_1(R,0,\dots,0)$.

Finally we define the point P by

$$P = \begin{cases} (\sqrt{2Ra+a^2},0,0,\dots,0),\ a\in \left(0,\frac{R}{2k(k+1)}\right] \equiv I, \\ (\frac{R}{k}-a,0,0,\dots,0),\ a\in \left[\frac{R}{2k(k-1)},\frac{R}{k}\right] \equiv J. \end{cases}$$

For the function $f(X) \equiv \omega(\rho(P,A_1)-\rho(P,X))$ we have

a) if $a\in I$, then

$$\|f(\cdot)-L(f;\cdot)\|_{C(T)} = |(f-L)(\sqrt{2Ra+a^2},0,0,\dots,0)|$$

$$= k+1-\frac{\sqrt{2Ra+a^2}}{R} = \varphi(a)$$

b) if $a\in J$, then

$$\|f(\cdot)-L(f;\cdot)\|_{C(T)} = |(f-L)(\frac{R}{k}-a,0,0,\dots,0)|$$

$$= k+\frac{(\sqrt{R^2-(\frac{R}{k}-a)^2}-R)(Rk-R+ka)}{akR} = \psi(a).$$

Let

$$\|f(\cdot)-L(f;\cdot)\|_{C(T)} \equiv \mathfrak{F}(a) \equiv \begin{cases} \varphi(a), & a\in I, \\ \psi(a), & a\in J. \end{cases}$$

The function $\mathfrak{F}(\cdot)$ is continuous, decreasing and $\lim \mathfrak{F}(a) = k+1$ $(a\to+\infty)$, $\mathfrak{F}(\frac{R}{k}) = k$, $(R=\frac{d_T}{2})$.

Thus, for any $\alpha^* \in [k,k+1)$ there is a number $a^* \in (0,\frac{R}{k})$ for which $\mathfrak{F}(a^*) = \alpha^*$.

References

1. Малоземов В.Н., Об отклонении ломеных. В.Ленингр. унив. , № 7, 1966, 150-153.
2. Логинов А.С. Приближение непрерывных функции ломаными. Мат. заметки, 6, 1969, № 2, 149-169.
3. Сендов Бл. Некоторые вопросы приближения функций в хаусдорфовой метрике. Успехи мат. наук., 24, 1969, № 5, 141-178.
4. Тотков Г.А.,М.С.Базелков, Об оптимальном полигональном интерполировании функций двух переменных, Доклады БАН, 33, 1980, № 5 603-605.
5. Тотков Г.А., М.С.Базелков, Метрические свойства модуля непрерывности и интерполяция сплайнами на треугольных сетках, Констр. теория функций '81, Тр. межд.конф. София, 1983, 171-177.
6. Тотков Г.А., М.С.Базелков, О точности полигональной интерполяции в H_{ω}^{s} ($s = 1,2$)

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