CONSTRUCTIVE THEORY OF FUNCTIONS'87

SOFIA, 1988

FINITE DIMENSIONAL APPROXIMATION FOR A CLASS OF OPTIMAL CONTROL PROBLEMS FOR SYSTEMS GOVERNED BY PARTIAL DIFFERENTIAL EQUATIONS

Tsvetomir Tsachev

1. Introduction. We shall consider optimal control problems connected with the boundary-value problems (B.V.P.) for the heat equation in a rectangle.

Let $\ell>0$, $T>0$, $Q_T=(0,\ell)\times(0,T)$, $f\in L_2(Q_T)$, $\varphi\in L_2(0,\ell)$ and $\alpha\geq 0$, $\beta\geq 0$.

Let us denote by $u(x,t;\varphi;f)$ the generalized solution of the following B.V.P. (first B.V.P.) (cf., e.g. [1], p. 392)

$$u_t-u_{xx}=f(x,t) \qquad \text{for } (x,t)\in Q_T \tag{1}$$

$$u(0,t)=u(\ell,t)=0 \qquad \text{for } t\in(0,T) \tag{2}$$

$$u(x,0;\varphi;f)=\varphi(x) \qquad \text{for } x\in(0,\ell), \tag{3}$$

or the generalized solution of the B.V.P. (third B.V.P.) (cf., e.g. [1], p. 393)

$$u_t-u_{xx}=f(x,t) \qquad \text{for } (x,t)\in Q_T \tag{1}$$

$$-u_x(0,t)+\alpha.u(0,t)=u_x(\ell,t)+\beta.u(\ell,t)=0 \qquad \text{for } t\in(0,T) \tag{2'}$$

$$u(x,0;\varphi;f)=\varphi(x) \qquad \text{for } x\in(0,\ell). \tag{3}$$

When $\alpha=\beta=0$ (in (2')) the problem (1), (2'), (3) is called second B.V.P.

Let us denote by $v_k(x)$ and λ_k the eigenfunctions and the eigenvalues of the problem

$$\begin{gathered} v''(x)-\lambda.v(x)=0 \\ v(0)=v(\ell)=0, \end{gathered} \tag{4}$$

or of the problem

$$\begin{gathered} v''(x)-\lambda.v(x)=0 \\ -v'(0)+\alpha.v(0)=v'(\ell)+\beta.v(\ell)=0. \end{gathered} \tag{5}$$

It is known that $\{v_k(x)\}_{k=1}^{\infty}$ is an orthonormal basis in $L_2(0,\ell)$. We further denote:

$$\varphi_k=(\varphi,v_k)_{L_2(0,\ell)} \quad ; \; f_k(t)=(f(x,t),v_k(x))_{L_2(0,\ell)};$$

$$U_k(t)=\varphi_k\cdot e^{\lambda_k t}+\int_0^t f_k(\tau)\cdot e^{\lambda_k(t-\tau)}d\tau.$$

The solution of the problem (1), (2), (3) or (1), (2'), (3) is given by

$$u(x,t;\varphi;f)=\sum_{k=1}^{\infty} U_k(t)\cdot v_k(x), \tag{6}$$

where $v_k(x)$ and λ_k in the formula for $U_k(t)$ are the eigenfunctions and the eigenvalues of (4) or (5) respectively (cf. [1], p. 399, Theorem 3).

The series in (6) is convergent in the norm of $H^{1,0}(Q_T)$ (cf. [1], p. 399, Theorem 3 and p. 166, §7.2). Using (6) it is easily shown that $u(x,t;\varphi;f)\in C[0,T;L_2(0,\ell)]$, i.e. for each $t\in[0,T]$ $u(\cdot,t;\varphi;f)\in L_2(0,\ell)$ and the mapping $u(\cdot,t;\varphi;f):[0,T]\longrightarrow L_2(0,\ell)$ (φ and f being fixed) is continuous.

<u>2. Statement of the problem and basic theorem.</u> Let $y(x)\in L_2(0,\ell)$ be given. Let us denote

$$J(\varphi,f)=\int_0^{\ell}|u(x,T;\varphi;f)-y(x)|^2dx=\|u(\cdot,T;\varphi;f)-y(\cdot)\|^2_{L_2(0,\ell)},$$

$$B_1=\{\psi\in L_2(0,\ell);\|\psi\|_{L_2(0,\ell)}\le 1\},\quad B_2=\{g\in L_2(Q_T);\|g\|_{L_2(Q_T)}\le 1\}.$$

We want to minimize $J(\varphi,f)$ over $B_1\times B_2$:

$$J(\varphi,f)\longrightarrow \min,\quad (\varphi,f)\in B_1\times B_2. \tag{P1}$$

The investigation of (P1) is motivated by [2]. The problem is studied there in the particular case when $f=0$ and, what is more important, without constraints on the initial condition φ. The constrained case is only briefly mentioned (p. 376). A problem similar to (P1) (in the particular case when $f=0$) has been studied in [4]. The type of constraints on φ is different there.

A crucial role in what follows will play

<u>Theorem 1.</u> Let E be an uniformly convex Banach space, $\{F_k\}_{k=1}^{\infty}$ be a sequence of closed convex subsets of E, $F_k\supset F_{k+1}$ for each natural k and $F=\bigcap_{k=1}^{\infty}F_k\neq\emptyset$. Let z be an arbitrary element of E and $\|z-x_k\|=\min_{x\in F_k}\|z-x\|$ for each natural k, $\|z-y\|=\min_{x\in F}\|z-x\|$. Then $\|x_k-y\|\xrightarrow[k\to\infty]{}0$.

3. Solving (P1) when $f(x,t)=0$ a.e. in Q_T. In this particular case the problem (P1) is reduced to

$$J(\varphi,0) \longrightarrow \min, \quad \varphi \in B_1 , \tag{P2}$$

i.e. we want to minimize $J(\varphi,0)$ over B_1.

When $f(x,t)=0$ a.e. in Q_T we have

$$u(x,t;\varphi;0) = \sum_{k=1}^{\infty} \varphi_k \cdot e^{\lambda_k t} \cdot v_k(x).$$

We define a linear operator $A: L_2(0,\ell) \longrightarrow L_2(0,\ell)$:

$$(A\varphi)(x) \overset{\text{def}}{=\!=} u(x,T;\varphi;0) = \sum_{k=1}^{\infty} \varphi_k \cdot e^{\lambda_k T} \cdot v_k(x) \quad (\varphi \in L_2(0,\ell)).$$

Then $J(\varphi,0) = \| A\varphi - y \|^2_{L_2(0,\ell)}$. It is known that A is compact.

Since $B_1 = \left\{ \varphi \in L_2(0,\ell);\ \sum_{k=1}^{\infty} (\varphi, v_k)^2_{L_2(0,\ell)} \le 1 \right\}$, it follows immediately that

$$A(B_1) = \left\{ \psi \in L_2(0,\ell);\ \sum_{k=1}^{\infty} \frac{(\psi, v_k)^2_{L_2(0,\ell)}}{e^{2\lambda_k T}} \le 1 \right\},$$

where $A(B_1) = \{ \psi \in L_2(0,\ell); \psi = A\varphi \text{ for } \varphi \in B_1 \}$ is the image of B_1.

The set $A(B_1)$ is closed convex subset of $L_2(0,\ell)$. Let us denote by $Pr_{A(B_1)} y$ the projection of y on the set $A(B_1)$, i.e.

$$\| Pr_{A(B_1)} y - y \|_{L_2(0,\ell)} = \min_{z \in A(B_1)} \| z - y \|_{L_2(0,\ell)}.$$

Let $\varphi_0 \in B_1$ be such that $A\varphi_0 = Pr_{A(B_1)} y$. Then

$$J(\varphi_0,0) = \| A\varphi_0 - y \|^2_{L_2(0,\ell)} = \| Pr_{A(B_1)} y - y \|^2_{L_2(0,\ell)} = \min_{z \in A(B_1)} \| z - y \|^2_{L_2(0,\ell)} =$$

$$= \min_{\varphi \in B_1} \| A\varphi - y \|^2_{L_2(0,\ell)} = \min_{\varphi \in B_1} J(\varphi,0).$$

Thus the problem (P2) is equivalent to the problem:

$$\text{find } Pr_{A(B_1)} y \tag{P2*}$$

We further denote $y_n = \sum_{k=1}^{n} (y, v_k)_{L_2(0,\ell)} \cdot v_k$ and $F_n = \left\{ \psi \in L_2(0,\ell);\ \sum_{k=1}^{n} \frac{(\psi, v_k)^2_{L_2(0,\ell)}}{e^{2\lambda_k T}} \le 1 \right\}$.

It is easily seen that

i) F_n is closed convex subset of $L_2(0,\ell)$ for each natural n;

ii) $F_n \supset F_{n+1}$ for each natural n;

iii) $A(B_1) = \bigcap_{n=1}^{\infty} F_n$.

Theorem 1 then yields $\| Pr_{F_n} y - Pr_{A(B_1)} y \|_{L_2(0,\ell)} \xrightarrow[n \to \infty]{} 0$.

Since $\|y_n - y\|_{L_2(0,\ell)} \xrightarrow[n\to\infty]{} 0$ and $\|Pr_K x - Pr_K y\|_X \le \|x-y\|_X$ for each $x, y \in X$ where X is a Hilbert space and K is a closed convex subset of X (cf., e.g. [3]), we see that

$$\begin{aligned}\|Pr_{F_n} y_n - Pr_{A(B_1)} y\|_{L_2(0,\ell)} &\le \|Pr_{F_n} y_n - Pr_{F_n} y\|_{L_2(0,\ell)} + \|Pr_{F_n} y - Pr_{A(B_1)} y\|_{L_2(0,\ell)} \le \\ &\le \|y_n - y\|_{L_2(0,\ell)} + \|Pr_{F_n} y - Pr_{A(B_1)} y\|_{L_2(0,\ell)} \xrightarrow[n\to\infty]{} 0.\end{aligned} \tag{7}$$

So we have

$$\|Pr_{F_n} y_n - y\|^2_{L_2(0,\ell)} \xrightarrow[n\to\infty]{} \|Pr_{A(B_1)} y - y\|^2_{L_2(0,\ell)} = \min_{\varphi \in B_1} J(\varphi, 0). \tag{8}$$

It is clear that $Pr_{F_n} y_n$ lies in the linear subspace spanned by $\{v_k\}_{k=1}^n$. Thus the problem of computing $Pr_{F_n} y_n$ is a finite-dimensional one and is formulated in the following way:

find the projection of the point $\left((y,v_1)_{L_2(0,\ell)}, \dots, (y,v_n)_{L_2(0,\ell)}\right) \in R^n$

on the ellipsoid $\tilde{F}_n = \left\{(x_1, x_2, \dots, x_n) \in R^n;\ \sum_{k=1}^{n} \frac{x_k^2}{e^{2\lambda_k T}} \le 1\right\} \subset R^n$.

The latter can be solved, say, using the Kuhn-Tucker theorem. By solving it, we obtain the coefficients $\zeta_{k,n}$ in the formula

$Pr_{F_n} y_n = \sum_{k=1}^{n} \zeta_{k,n} \cdot v_k$. Thus $\varphi^n = \sum_{k=1}^{n} \frac{\zeta_{k,n}}{e^{\lambda_k T}} \cdot v_k$ can serve as an "approximate" solution to the problem (P2), because (8) yields

$$J(\varphi^n, 0) = \|A\varphi^n - y\|^2_{L_2(0,\ell)} = \|Pr_{F_n} y_n - y\|^2_{L_2(0,\ell)} \xrightarrow[n\to\infty]{} \min_{\varphi \in B_1} J(\varphi, 0).$$

Thus the problem (P2*), which is an infinite dimensional one, has been approximated by finite-dimensional problems (find $Pr_{F_n} y_n$) in the sense that their solutions converge to the solution of (P2*) (see (7)).

4. Solving (P1) when $\varphi(x) = 0$ a.e. in $(0, \ell)$. In this particular case the problem (P1) is reduced to

$$J(0,f) \longrightarrow \min,\ f \in B_2, \tag{P3}$$

i.e. we want to minimize $J(0,f)$ over B_2.

When $\varphi(x) = 0$ a.e. in $(0,\ell)$ we have the expansion

$$u(x,t;0;f) = \sum_{k=1}^{\infty} \left\{\int_0^t f_k(\tau) \cdot e^{\lambda_k(t-\tau)} d\tau\right\} \cdot v_k(x).$$

We define a linear operator $\tilde{A}: L_2(Q_T) \to L_2(0,\ell)$:

$$(\tilde{A}f)(x) \overset{def}{=\!=} u(x,T;0;f) = \sum_{k=1}^{\infty} \left\{ \int_0^T f_k(\tau) e^{\lambda_k(T-\tau)} d\tau \right\} \cdot v_k(x).$$

It can be shown that $\tilde{A}$ is compact and that

$$\tilde{A}(B_2) = \left\{ \psi \in L_2(0,\ell); \sum_{k=1}^{\infty} \frac{(\psi, v_k)^2_{L_2(0,\ell)}}{\rho_k^2} \le 1 \right\}$$, where

$$\rho_k = \sqrt{\int_0^T e^{2\lambda_k(T-\tau)} d\tau}$$ and $$\tilde{A}(B_2) = \{ \psi \in L_2(0,\ell);\ \psi = \tilde{A}f \text{ for } f \in B_2 \}$$

is the image of B_2.

$\tilde{A}(B_2)$ is closed convex subset of $L_2(0,\ell)$. Hereafter, denoting

$$G_n = \left\{ \psi \in L_2(0,\ell);\ \sum_{k=1}^{n} \frac{(\psi, v_k)^2_{L_2(0,\ell)}}{\rho_k^2} \le 1 \right\}$$

we can proceed exactly as in the case of (P2).

5. Solving (P1) in the general case. It is clear that $u(x,T;\varphi;f) = A\varphi + \tilde{A}f \in L_2(0,\ell)$. Since $A(B_1)$ and $\tilde{A}(B_2)$ are convex compacts, $A(B_1)+\tilde{A}(B_2) = \{ \psi \in L_2(0,\ell)\ ;\ \psi = \psi_1 + \psi_2$, where $\psi_1 \in A(B_1), \psi_2 \in \tilde{A}(B_2) \}$ is a closed convex sbset of $L_2(0,\ell)$. Thus $Pr_{A(B_1)+\tilde{A}(B_2)} y$ is well defined and

$$\| Pr_{A(B_1)+\tilde{A}(B_2)} y - y \|^2_{L_2(0,\ell)} = \min_{(\varphi,f) \in B_1 \times B_2} J(\varphi, f) .$$

We can apply the technique used for solving (P2) and (P3) because of the following

Lemma 1. It is true that $A(B_1)+\tilde{A}(B_2) = \bigcap_{k=1}^{\infty} (F_k + G_k)$.

References

1. В.П.Михайлов, Дифференциальные уравнения в частных производных /второе издание/, Москва, "Наука", 1983

2. R. Lattes and J.-L. Lions. The method of quasi-reversibility. Modern Analitic and Computational Methods in Science and Mathematics. American Elsevier Publishing Company, Inc. New York, 1969

3. D. Kinderlehrer and G. Stampacchia. An introduction to variational inequalities and their applications. Academic press, New York-London-Toronto-Sydney-San Francisko, 1980

4. Ю. В. Егоров. Некоторые задачи теории оптимального управления Журнал вычислительной математики и математической физики, 1963, 3, № 5, стр. 887-904

Institute of Mathematics
Bulgarian Academy of Sciences
1090 Sofia Bulgaria