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ON MODULAR AND ORLICZ SPACE OVER A FIELD WITH VALUATION

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1. (w,v)-convex modulars. Let X be a vector space over a field with a non-trivial valuation $|\cdot|$. Given are mappings $w, v: \mathbb{K} \to \mathbb{R}_+$ ($\mathbb{R}_+ = [0,\infty)$) satisfying the following conditions:

(a1) There are constants $k, l > 0$ such that kw and lv are supermultipicative mappings on $\mathbb{K}$.

(a2) $w(a)\ v(a) > 0$ for $a \in \mathbb{K}_* = \mathbb{K} \setminus \{0\}$.

(a3) w is a continuous in 0 and $w(0) = 0$. . There exist $q > 0$ such that $\lim w(a)/v(a) \geq q$ as $|a| \to \infty$.

(a4) There exist an $M > 0$ such that for every $B > 0$ there exist $a, b \in \mathbb{K}_*$ such that $0 < w(a) \leq B \leq w(b) \leq Mw(a)$.

(a5) If $(a_n) \subset \mathbb{K}_*$ is a bounded sequence then $1/v(a_n^{-1})$ and $1/w(a_n^{-1})$ are bounded.

A functional $m: X \to [0,\infty]$ is called (w,v)-convex modular if the following conditions are satisfied:

(m1) $m(0) = 0$, and $m(ax) = 0$ for $a \in K_*$ implies $x = 0$.

(m2) $m(x) = m(-x)$.

(m3) $m(ax + by) \leq v(a)\ m(x) + v(b) m(y)$ for all $a, b \in \mathbb{K}$ with $w(a) + w(b) \leq 1$.

The vector space $X_m = \{x \in X: \lim m(ax) = 0 \text{ as } a \to 0\}$ is called the modular space generated by m. If m is a (w,v)-convex modular on X, then the functional $\|x\|_m = \inf\{w(a): a \in \mathbb{K}_* \text{ and } m(x/a)\,v(a) \leq w(a)\}$ is an F-quasinorm on X_m with the constant $C = M \min^{-1}(k,l)$ in the triangle inequality (see [4, 5]). In particular given $w(a) = k^{-1}|a|^s$, $v(a) = l^{-1}|a|^t$ $s \geq t \geq 0$, $s > 0$ we obtain all the types of the classical modulars. For example convex modulars, nonconvex, p-convex and Nakano modulars [1, 2].

2. A representation of supermultipicative function. In this part we restrict ourselves to the real case i.e. $\mathbb{K} = \mathbb{R}$. We say that the function $w: \mathbb{R}_+ \to \mathbb{R}_+$ is submultipicative [supermultipicative] if $w(ab) \leq w(a)w(b)$ [resp. $w(ab) \geq w(a)w(b)$] for all $a, b \in \mathbb{R}_+$.

Lemma 1. If the function $f\colon \mathbb{R}\to\mathbb{R}$ is subadditive or superadditive and has a derivative at zero, $f'(0)=c$, $f(0)=0$, then $f(a)=ca$.

Proof. We observe that $f(na)\leq nf(a)$. Hence $f(a)\leq nf(a/n) = af(a/n)/(a/n)$, and we obtain $f(a)\leq ca$. But $f(0)\leq f(a)+f(-a)\leq f(a)+nf(-a/n)$. Hence $f(a)\geq af((-a/n))/(-a/n)$, consequently $f(a)\geq ca$. If f is superadditive we given $g(a)=-f(a)$.

Proposition 1. If $w\colon \mathbb{R}_+\to\mathbb{R}_+$ and kw is submultipicative or supermultipicative with some constants $k>0$. Moreover if w has a derivative at 1 such that $w'(1)=k^{-1}s$ and $w(1)=k^{-1}$, then $w(a)=k^{-1}a^s$.

Proof. Given the function $f(b)=\log kw(e^b)$, we have $f(0)=0$, $f'(0)=s$. Moreover f is subadditive or superadditive if w is submultipicative or respectively supermultipicative. Hence by the Lemma $f(b)=sb$. Taking $a=e^b$ we get $w(a)=k^{-1}a^s$.

From the Lemma 1 it follows that every subadditive function such that $f(0)=0$, $f'(0)=0$ is linear. Hence it is positive homogeneous. Now we show the following.

Lemma 2. If the function $f\colon \mathbb{R}\to\mathbb{R}$ is a p-positive homogeneous $(0<p\leq 1)$, then $f(a)=(c\operatorname{sign} a+d)|a|^p$, for some real numbers c,d.

Proof. We observe that $f(a)=|a|^p f(\operatorname{sign} a)$. Now taking $2c=f(1)-f(-1)$, $2d=f(1)+f(-1)$ we have $f(a)=(c\operatorname{sign} a+d)|a|^p$.

Lemma 3. If the function $f\colon \mathbb{R}\to\mathbb{R}$ is a p-positive homogeneous $(0<p\leq 1)$ and subadditive, then $f(a)=(c\operatorname{sign} a+d)|a|^p$, where $d\geq |c|(1-[p])$, and [p] denote the integral part of p.

Proof. In the case $p=1$ we have $f(a)=ca+d|a|$. Hence $d\geq 0$. Now, let $0<p<1$ by the Lemma 2, $f(a)=(c\operatorname{sign} a+d)|a|^p$. Given any $a,b\in\mathbb{R}$, we consider the following cases

(i) $\operatorname{sign} a=\operatorname{sign} b$, then $f(a+b)\leq f(a)+f(b)$ iff $(c\operatorname{sign} a+d)|a+b|^p\leq (c\operatorname{sign} a+d)|a|^p+(c\operatorname{sign} a+d)|b|^p$. Hence $c+d\geq 0$ and $-c+d\geq 0$ this is equivalent $d\geq |c|$.

(ii) Now let $d\geq |c|$, and $\operatorname{sign} a=\operatorname{sign}(a+b)=-\operatorname{sign} b$. Then $a+b=\operatorname{sign} a|a+b| = \operatorname{sign} a|a| - \operatorname{sign} a|b|$ and we have $|a+b|=|a|-|b|$. This implies $|a+b|\leq |a|$. Hence $(c\operatorname{sign} a+d)|a+b|^p\leq (c\operatorname{sign} a+d)|a|^p$, finally $f(a+b)\leq f(a)+f(b)$.

Corollary 1. If the function $f\colon \mathbb{R} \to \mathbb{R}$ is a p-positive homogeneous $(0 < p \leq 1)$ and superadditive, then $f(a) = (c \operatorname{sign} a + d)|a|^p$, where $d \leq |c|([p] - 1)$.

Proof. We observe that the function $g(a) = -f(a)$ is p- homogeneous and subadditive. Hence by the Lemma 3, we get $g(a) = (c_1 \operatorname{sign} a + d_1)|a|^p$, where $d_1 \geq |c_1| (1 - [p])$. Denote $c = -c_1$, $d = -d_1$ we have $f(a) = (c \operatorname{sign} a + d)|a|^p$, where $d \leq |c| ([p] - 1)$.

We say that the function $w\colon \mathbb{R}_+ \to \mathbb{R}_+$ satisfies the conditions (H) with the constant $k > 0$ and $0 < p \leq 1$ if for $h > 0$

(H) $w(a^h) = k^{1-h^p} w^{h^p}(a)$, for all $a \geq 0$.

Proposition 2. If the function $w\colon \mathbb{R}_+ \to \mathbb{R}_+$ satisfies the condition (H) with the constant $k > 0$, $0 < p \leq 1$, and kw is submultipicative, then

$w(a) = k^{-1}\exp(s \operatorname{sign}(\log a) + t)|\log a|^p$, where $t \geq |s| (1 - [p])$.

Proof. Given $f(b) = \log k^{-1} w(\exp b)$. The function f is subadditive and p-positive homogeneous. Hence by Lemma 3, we obtain $f(b) = (s \operatorname{sign} b + t)|b|^p$, where $t \geq |s| (1 - [p])$. Given $b = \log a$, we have $w(a) = k^{-1}\exp(s \operatorname{sign}(\log a) + t)|\log a|^p$.

Corollary 2. If the function $w\colon \mathbb{R}_+ \to \mathbb{R}_+$ satisfies the condition (H) with the constant $k > 0$, $p = 1$ and kw is submultipicative, then

$$w(a) = \begin{cases} k^{-1} a^{s+t}, & \text{for } a \geq 1, \\ k^{-1} a^{s-t}, & \text{for } 0 \leq a < 1, \end{cases} \quad \text{where } t \geq 0.$$

Corollary 3. If the function $w\colon \mathbb{R}_+ \to \mathbb{R}_+$ satisfies the condition (H) with the constants $k > 0$, $p = 1$ and kw is supermultipicative, then

$$w(a) = \begin{cases} k^{-1} a^{s+t}, & \text{for } a \geq 1, \\ k^{-1} a^{s-t}, & \text{for } 0 \leq a < 1, \end{cases} \quad \text{where } t \geq 0.$$

3. Existence of nontrivial additive functionals on nonarchimedean modular space. In this section we consider a field $\mathbb{K}$ with non-trivial valuation $|\cdot|$, and we take $w(a) = k^{-1}|a|^s$, $v(a) = 1^{-1}|a|^t$, $s \geq t \geq 0$, $s > 0$. In this case for simplicity the (w,v)-convex modular on X will be called modular. Let (x_n) be a sequence of elements of the modular space X_m ; (x_n) is called modular convergent m-convergent to x if there exists $b \neq 0$ such that $m(b(x_n - x)) \to 0$ as $n \to \infty$; we write

$x_n \xrightarrow{m} x$ (see [1]). Convergence in F-quasinorm in X_m implies m-convergence. A functional x^* over X_m is called modular continuous if $x_n \xrightarrow{m} x$ implies $x^*(x_n) \to x^*(x)$ for any $x \in X_m$. A modular m satisfies the condition D_2 (see [2]) if has the following property: for every $x \in X$ there exists $y \in X$ such that $m(x - y) \leq m(x)/2$ and $m(y) \leq m(x)/2$. A modular m satisfies the condition B_2 if $m(x_n) \to 0$ implies $m(ax_n) \to 0$ for every $a \in \mathbb{K}$.

Theorem. Let K be a nonarchimedean valued field and m be a modular, satisfying the condition D_2. Then there exists no nontrivial modular continuous additive functional on the modular space X_m.

Proof. Suppose that there exists a nontrivial additive functional x^* on X_m. Hence there exists $x_0 \in X_m$ and $a \neq 0$ such that $x^*(x_0) \neq 0$, $m(ax_0) < \infty$. Denote $y^*(x) = (x(x_0))^{-1} x^*(a^{-1}x)$, $y_0 = ax_0$, then $y^*(y_0) = 1$, $m(y_0) < \infty$ and y^* is additive and modular continuous functional on X_m. From the condition D_2 follows the existence of $y_1, z_1 \in X_m$ such that $y_0 = y_1 + z_1$, $m(y_1) \leq m(y_0)/2$ and $m(z_1) \leq m(y_0)/2$. But $1 = |y^*(y_0)| \leq \max(|y^*(y_1)|, |y^*(z_1)|)$. Hence we can find a sequence $(y_n) \subset X_m$ such that $1 \leq |y^*(y_n)|$ and $m(y_n) \to 0$, and we obtain a contradiction.

If modular m satisfies the condition B_2 then modular convergence is equivalent to F-quasinorm convergence. Hence we obtain

Corollary 1. Let X be a nonarchimedean valued field and m be a modular satisfying the condition B_2 and D_2. Then there exists no nontrivial continuous additive functional on the modular space X_m.

Let μ be a positive measure defined on a σ-field Σ of subsets of a set S and let E be a Banach space over a field $\mathbb{K}$ with valuation $|\cdot|$. Let $\mathcal{M}$ denote the linear space of all μ-measurable function $x: S \to E$ modulo the functions which are zero almost everywhere. Let $\phi : [0,\infty) \to [0,\infty]$ be a nondecreasing function vanishing only at zero, continuous from the left on $\mathbb{R}_+$ and continuous at 0. The functional $I = I_\phi : \mathcal{M} \to [0,\infty]$ defined by $I_\phi(x) = \int_S \phi(\|x\|)\, d\mu$ is a modular on $\mathcal{M}$. The modular space $\mathcal{M}_I$ is called the Orlicz space and is denoted by L^ϕ ([1, 3]) The functional $\|x\|_\phi = \inf\{|a| : I_\phi(x/a) \leq |a|, a \neq 0\}$ is an F-quasinorm on L^ϕ with constant $C = r$, where $r = \inf\{|a| > 1 : a \in \mathbb{K}\}$. The function ϕ satisfies the condition Δ_2 if there are two constants $a > 0$, $b > 0$ such that $\phi(2c) \leq b\phi(c)$ for every $c \geq a$.

Corollary 2. Let $\mathbb{K}$ be a nonarchimedean valued field. Let ϕ satisfy condition Δ_2 and let μ be a finite nonatomic measure. Then there exists no nontrivial continuous additive functionals on the Orlicz space L^{ϕ}.

The above Theorem are generalization of the results proved in [5]. In the classical case, i.e. for an archimedean valued field the above theorem is not true, [3] .

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