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BEST APPROXIMATIONS OF SOLUTIONS OF SINGULAR INTEGRAL EQUATIONS OF FIRST KIND

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1. Introduction. In many application problems is found the singular integral equation of the form :

$$/1/ \qquad Kx \equiv \frac{1}{\pi}\int_{-1}^{+1}\frac{x(\tau)d\tau}{(\tau-t)\sqrt{1-\tau^2}} + \frac{1}{\pi}\int_{-1}^{+1}\frac{h(t,\tau)x(\tau)d\tau}{\sqrt{1-\tau^2}} = y(t) ,$$

$$/2/ \qquad \int_{-1}^{+1}\frac{x(\tau)d\tau}{\sqrt{1-\tau^2}} = 0 ,$$

where $h(t,\tau)$ and $y(t)$ are given functions from a certain class $F=\{f\}$, $x(\tau)$ is the requested function, and the singular integral in /1/ is understood in the sense of principal value in Cauchy - Lebesgue.

Equations /1/-/2/ are juxtaposed in accordance with the system of linear algebraic equations /SLAE/ :

$$/3/ \qquad \sum_{k=0}^{n} A_k(t_j)c_k + \sum_{k=0}^{n} B_k h(t_j,\tau_k)c_k = y(t_j), \ j=\overline{1,n} ,$$

$$/4/ \qquad \sum_{k=0}^{n} B_k c_k = 0 ,$$

where $A_k=A_{k,n}(t)$, $B_k=B_{k,n}$ are coefficients, $\tau_k=\tau_{k,n}\in[-1,1]$, $k=\overline{0,n}$ - nodes of the used quadrature formulae, $t_j=t_{j,n}\in(-1,1)$, $j=\overline{1,n}$ - are collocation nodes, $c_k=c_{k,n}$ - approximate value of the requested function $x(t)$ in the nodes τ_k.

The following function is assumed for the approximate solution of the equations /1/-/2/ :

$$/5/ \qquad x_n(t)=m_n(\{c_k\};t)=M_n(h,y;\{A_k,B_k,\tau_k,t_j\};t) ,$$

where $\{c_k\}_0^n$ is the solution of the system /3/-/4/, m_n and M_n are certain reconstruction operators of a rank not greater than n. Then the following quantity is used for an optimal error estimate of the class of quadrature methods /3/-/4/ :

/6/ $$V_n(F)=\inf \inf_{A_k,B_k,\tau_k,t_j} \sup_{h,y\in F} \|x-x_n\|_U ,$$

where the outer inf is taken by an arbitrary reconstruction operator of a rank not greater than n, U is a certain normed space, which contains the elements $x(t)$ and $x_n(t)$. The fixed quadrature method of the form /3/-/5/, for which the quantity /6/ is reached at least by the order, is called optimal.

In the article have been calculated the optimal error estimates /6/ for different function classes and for the solutions /1/-/2/ have been established effective optimal quadrature methods of the form /3/-/5/.

2. Main Results. With $W^rH^\omega[-1,1]$ we denote, as usual, the class of r-times /$r\geqq 0$ - an integer/ continuously differentiable functions in the interval $[-1,1]$, the continuity moduli of the r-th derivates which do not exceed the preset continuity modulus $\omega=\omega(\delta)$, $0<\delta\leqq 2$. Let $F=\{f\}$ is a set of couples $f=\{h,y\}$, where $h(t,\tau)\in W^rH^\omega[-1,1]$ /by each of the arguments separately evenly with regard to the other one of them/ and $y(t)\in W^rH^\omega[-1,1]$, and on $h(t,\tau)$, beside that, is imposed the term :

/7/ $$\|K\|_{X\to Y}\leqq c_0<\infty \quad , \|K^{-1}\|_{Y\to X}\leqq c_1<\infty ,$$

where c_0 and c_1 are certain positive constants, which are common for the whole class F ; furthermore :

$$X=\overset{0}{L}_{2,\rho}=\left\{x\in L_{2,\rho}(-1,1): \int_{-1}^{+1}\rho(\tau)x(\tau)d\tau=0\right. \quad , Y=L_{2,\rho^{-1}}(-1,1)$$

is a space of quadratically summable functions in $[-1,1]$ with weights correspondingly $\rho(t)=(1-t^2)^{-1/2}$, $\rho^{-1}(t)=(1-t^2)^{1/2}$ and with usual norms.

For the sake of brevity we denote the class F below as follows : $F=W^rH^\omega$.

There exist the following theorems.

Theorem 1. Let $F=W^rH^\omega$ and $U=L_{2,\rho}(-1,1)$. Then

/8/ $$V_n(F)\asymp\frac{1}{n^r}\omega\left(\frac{1}{n}\right),$$

and optimal is the method of the mechanical quadratures with approximate solution :

/9/ $$x_n(t)=\frac{\cos(n+1)\arccos t}{n+1}\sum_{k=0}^{n}(-1)^k c_k\frac{\sin\theta_k}{t-\cos\theta_k} \quad , \theta_k=\frac{2k+1}{2n+2}$$

where $\{c_k\}_0^n$ is the solution of SLAE :

/10/ $$\frac{1}{n+1}\sum_{k=0}^{n} c_k\left[\frac{1}{\tau_k - t_j} + h(t_j,\tau_k)\right] = y(t_j), \quad j=\overline{1,n} \quad ,$$

/11/ $$\sum_{k=0}^{n} c_k=0, \quad \tau_k=\cos\theta_k, \quad t_j=\cos\varphi_j, \quad \varphi_j=\frac{j\pi}{n+1} \ .$$

Theorem 2. Let $F=W^rH^\omega$ and $U=C[-1,1]$, where for $r=0$ we suppose the term :

/12/ $$\lim_{\delta\to+0}\omega(\delta)\ln\delta=0.$$

Then

/13/ $$V_n(F)\asymp\frac{\ln n}{n^r}\,\omega\left(\frac{1}{n}\right) \quad ,$$

and the method of the mechanical quadratures, determined by the formulae /9/-/11/, is optimal, where $\asymp$ denotes sign of poor equivalence.

3. About the Proof of Theorems 1 and 2. In first place the problem /1/-/2/ is written in the form of an equivalent operator equation, which is resolvable in a unique manner, of the form :

/14/ $$Kx \equiv Sx + Thx = y \qquad (x\in X,\ y\in Y),$$

where the operators S and T are determined through the formulae :

$$Sx=\frac{1}{\pi}\int_{-1}^{+1}\frac{x(\tau)d\tau}{(\tau-t)\sqrt{1-\tau^2}} \quad , \quad Thx=\frac{1}{\pi}\int_{-1}^{+1}\frac{h(t,\tau)x(\tau)d\tau}{\sqrt{1-\tau^2}} \ .$$

It is known (e.g. [1]) that $S:X\to Y$ - continuous and $T:X\to Y$ - fully continuous operators and besides the operator S has continuous opposite $S^{-1}:Y\to X$.

With H_m we denote the set of all algebraic polynomials of order not greater than m and we introduce n-dimensional subspaces of the spaces X and Y correspondingly:

$$X_n=\overset{\circ}{H}_n=\left\{x_n\in H_n : \int_{-1}^{+1}\rho(t)x_n(t)dt=0\right\} \ , \ Y_n=H_{n-1} \ .$$

With $T_m(t)$ and $U_m(t)$ we denote Chebyshev polynomials of order m correspondingly of first and second kind and we introduce the operators :

$$P_n^U:C[-1,1]\to H_{n-1} \ , \ P_{n+1}^T:C[-1,1]\to H_n \ ,$$

putting

$$P_n^U\varphi(t)=\sum_{k=0}^{n}\varphi(t_k)l_{k,n}^U(t), \ U_n(t_k)=0, \ \varphi\in C[-1,1] \quad ,$$

$$P_{n+1}^{T}\varphi(t)=\sum_{k=0}^{n}\varphi(\tau_k)l_{k,n+1}^{T}(t),\quad T_{n+1}(\tau_k)=0,\ \varphi\in C[-1,1]\quad;$$

here $l_{k,n}^{U}(t)$ and $l_{k,n+1}^{T}(t)$ are fundamental Lagrangial polynomials in the nodes correspondingly

$$t_k=t_{k,n}^{U}=\cos\frac{k\pi}{n+1},\quad k=\overline{1,n}\quad;\quad \tau_k=\tau_{k,n+1}^{T}=\cos\frac{2k+1}{2n+2}\pi,\quad k=\overline{0,n}\ .$$

It is known that for arbitrary $\varphi\in C[-1,1]$

/15/ $\|\varphi-P_n^U\varphi\|_Y\leqq\sqrt{2\pi}\ E_{n-1}(\varphi)_C\ ,\ n\in N\ ,$

/16/ $\|P_n^U\varphi\|_Y\leqq\sqrt{\dfrac{\pi}{2}}\,\|\varphi\|_C\ ,\quad n\in N\ ,$

/17/ $\|\varphi-P_{n+1}^T\varphi\|_X\leqq 2\sqrt{\pi}\ E_n(\varphi)_C\quad .$

An arbitrary polynomial $x_n\in X_n$ can be expressed in the form :

$$x_n(t)=\sum_{k=1}^{n}\alpha_k T_k(t)=\sum_{k=0}^{n}x_n(\tau_k)l_{k,n+1}^{T}(t)\ ,$$

and that is why the term /2/ is equivalent to the term

/2'/ $\displaystyle\sum_{k=0}^{n}x_n(\tau_k)=0,\quad n\in N.$

From here with taking into account the relations

$$\alpha_k=\frac{2}{\pi}\int_{-1}^{+1}\frac{x_n(\tau)T_k(\tau)d\tau}{\sqrt{1-\tau^2}}\ ,\quad l_{k,n+1}^{T}(t)=\frac{T_{n+1}(t)}{(t-\tau_k)T'_{n+1}(\tau_k)}$$

and the properties of the singular operator S (e.g. [1,3,4]) we have

$$S(x_n;t_j)=\sum_{k=1}^{n}\alpha_k U_{k-1}(t_j)=\frac{1}{n+1}\sum_{k=0}^{n}\frac{x_n(\tau_k)}{t_j-\tau_k}\ ,\quad j=\overline{1,n}\quad .$$

Then (see [2,4,6]) SLAE /10/-/11/ is equivalent to the operator equation

/18/ $\quad K_n x_n\equiv P_n^U S x_n + P_n^U T P_{n+1}^T(hx_n) = P_n^U y(x_n\in X_n, P_n^U y\in Y_n)\ ,$

where $K_n:X_n\longrightarrow Y_n$ is a linear operator by each natural n and the operator P_{n+1}^T is applied to the function $h(t,\tau)x_n(\tau)$ by the variable τ .

Since $Sx_n\in Y_n$ for arbitrary $x_n\in X_n$ and the quadrature formula of Gauss with nodes $\tau_k=\tau_{k,n+1}^T$, $k=\overline{0,n}$, is exact for all algebraic polynomials of order not greater than 2n+1, then the equation /18/ is equivalent to the operator equation

/19/ $$K_n x_n \equiv Sx_n + P_n^U T\left[(P_{n+1}^T h)x_n\right] = P_n^U y \quad (x_n \in X_n, P_n^U y \in Y_n).$$

Now from /14/ and /19/ for arbitrary $x_n \in X_n$ we find

$$\| Kx_n - K_n x_n \|_Y = \| Thx_n - P_n^U T(P_{n+1}^T h)x_n \|_Y \leq \rho_1 + \rho_2 ,$$

where

$$\rho_1 = \| Thx_n - P_n^U Thx_n \|_Y , \quad \rho_2 = \| P_n^U T\left[(h - P_{n+1}^T h)x_n\right] \|_Y .$$

From here, as in the article [4] with taking into account the relations /15/-/17/ for arbitrary $x_n \in X_n$ we find the inequality

$$\rho_1 \leqq d_1 E_{n-1}^t(h)_C \|x_n\|_X , \quad \rho_2 \leqq d_2 E_n^\tau(h)_C \|x_n\|_X ,$$

where $E_m^t(h)_C$ ($E_m^\tau(h)_C$) is a partial best even approximation of the function $h(t,\tau)$ with algebraic polynomials of order m by the variable t (correspondingly τ) evenly with regard to τ (with regard to t) and d_i (i=1,2,...) here and after are positive constants which do not depend on n (and also on $x_n \in X_n$). Therefore, we have

/20/ $$\varepsilon_n \equiv \| K - K_n \|_{X_n \to Y} \leqq d_3 \left\{ E_{n-1}^t(h)_C + E_n^\tau(h)_C \right\} , \quad n \in N ,$$

where $d_3 = \max(d_1, d_2)$. Besides, for the right side of the equations /14/ and /19/ because of /15/ we have

/21/ $$\delta_n \equiv \| y - P_n^U y \|_Y \leqq \sqrt{2\pi}\, E_{n-1}(y)_C , \quad n \in N.$$

In this manner for the equations /14/ and /19/ have been satisfied all terms of theorem 7, chapter I of the general theory of the analysis approximate methods [2] . This is the reason for all n such as

$$q_n = \|K^{-1}\| \varepsilon_n \leqq c_1 d_3 \left\{ E_{n-1}^t(h)_C + E_n^\tau(h)_C \right\} < 1 ,$$

equation /19/, and hence, and SLAE /10/-/11/ are resolvable in a unique manner and besides

$$\| K_n^{-1} \| \leqq \|K^{-1}\| (1 - q_n)^{-1} \leqq 2\|K^{-1}\| \leqq 2c_1 , \quad n \geqq n_0 .$$

Besides, due to the same theorem, the approximate solution of /9/, where $\{c_k\}_0^n$ is the solution of the system /10/-/11/, is converging to the exact solution x(t) of the problems /1/-/2/ with convergence rate

$$\| x - x_n \|_X \leqq \frac{\|K^{-1}\|}{1 - q_n} \left[\| y - P_n y \|_Y + q_n \|y\|_Y \right] =$$

/22/ $$= O\left\{ E_{n-1}^{t}(h)_C + E_n^{\tau}(h)_C + E_{n-1}(y)_C \right\} .$$

From here and from Jackson's theorem and taking into account the terms of theorem 1 there foloows the estimate :

/23/ $$\sup_{h,y \in F} \|x-x_n\|_X = O\left\{ \frac{1}{n^r} \omega\left(\frac{1}{n}\right) \right\} ,$$

It is enough to show the validity of the estimate in formula /8/ for completeing the proof of theorem 1. For this purpose we introduce the set $F_o=\{f_o\}$, where $f_o=(0;y)$. It is obvious that $F_o \subset F$ and hence $V_n(F_o) \leqq V_n(F)$. Since by term y $W^r H^\omega$ and the operator S is linearly reversible, then from chapter 3 [2] there follows the requested estimate

/24/ $$V_n(F_o) \geqq \frac{d_4}{n^r} \omega\left(\frac{1}{n}\right) .$$

In this manner the theorem 1 has been fully proven.

We notice, when moving to the proof of theorem 2, that the lower estimate in /13/ is drawn again by means of chapter 3 from monograph [2] and the article [5] and it is shown that

/25/ $$V_n(F) \geqq V_n(F_o) \geqq \frac{d_5 \ln n}{n^r} \omega\left(\frac{1}{n}\right) .$$

Then for the completion the proof of the theorem 2 there remains to be shown that

/26/ $$\sup_{h,y \in F} \|x-x_n\|_C = O\left\{ \frac{\ln n}{n^r} \omega\left(\frac{1}{n}\right) \right\} ,$$

where x(t) is the exact solution of the problem /1/-/2/, and $x_n(t)$ is its approximate solution, established by the formula /8/. With that purpose we deal with the identity

$$Sx \equiv y - Thx ,$$

$$Sx_n \equiv P_n^U y - P_n^U T\left[(P_{n+1}^T h)x_n\right] ,$$

where x and x_n are solutions of the equations /14/ and /19/ correspondingly. From here follows the identity

/27/ $$S(x-x_n) = (Sx - P_n^U Sx) + P_n^U S(x-x_n) .$$

Then

/28/ $$\|x-x_n\|_C \leqq \|S^{-1}(Sx-P_n^U Sx)\|_C + \|S^{-1} P_n^U S(x-x_n)\|_C$$

$$\leqq \|x-S^{-1}P_n^U Sx\| + \| S^{-1}P_n^U T\left[hx-P_{n+1}^T (hx_n)\right]\|_C .$$

Now from /28/ with taking into account the properties of the operators S, S^{-1}, P_n^U and the corresponding results of chapter 3 [2] there follows the estimate /26/. In this case, as in the solution of an analogous problem in article [4] , essentialy is used theorem 1.

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