CONSTRUCTIVE THEORY OF FUNCTIONS'87

SOFIA, 1988

SETS OF APPROXIMABLE ELEMENTS IN DOUBLE SEQUENCE SPACES

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1. Definitions and preliminaries. Starting point for this note are papers of A.Alexiewicz and W.Orlicz [1], [2], of J.Musielak [3], [4] and also papers [6], [7], [9] and [11]. We consider an example of modular space connected with strong summability of double sequences. For a given sequence of such spaces we introduce some sets of approximable elements. Properties of these sets and connections between them are main result of this note.

Let T denote space of all real double sequences in which convergence we shall mean as convergence in the sense of Pringsheim. Sequences belonging to T will be denoted by $x=(t_{\mu\nu})$, $y=(s_{\mu\nu}),\ldots$, $|x|=(|t_{\mu\nu}|)$, $\Theta=(0)$ and $x^j=(t^j_{\mu\nu})$ for $j=1,2,\ldots$. If $x^1,x^2\in T$, the inequality $x^1\geq x^2$ will mean $t^1_{\mu\nu}\geq t^2_{\mu\nu}$ for $\mu,\nu=1,2,\ldots$.

Let T_0, T_f and T_b denote spaces of all real double sequences convergent to zero, double seguences with a finite number of elements different from zero and bounded real double sequences, respectively.

A function φ defined in the interval $[0,\infty)$, nondecreasing and continuous for $u\geq 0$ and such that $\varphi(0)=0$, $\varphi(u)>0$ for $u>0$, $\varphi(u)\to\infty$ as $u\to\infty$, is called a φ-function. φ-functions will be denoted by $\varphi,\varphi_1,\varphi_2,\ldots$. Let (φ_i) denote sequence of φ-functions.

In the sequel the following conditions will be often used:

(1) $\varphi_i(u)$ are equicontinuous at $u=0$,

(2) there exists a number u_0 such that $\varphi_1(u_0)+\varphi_2(u_0)+\ldots<\infty$,

(3) there exists a $u'>0$ such that $\varphi_1(u')+\varphi_2(u')+\ldots=\infty$,

(4) there exists a $\underline{u}>0$ such that $\sup_i \varphi_i(\underline{u})=\infty$,

(5) for every index i there exist positive constants λ_i, β_i, ν_i such

that for every $u \leq v_i$ and $k \geq i$ the inequality $\varphi_i(\lambda_i u) \leq \beta_i \varphi_k(u)$ holds,

(6) for every $\varepsilon > 0$ there exist numbers $u_\varepsilon > 0$ and $\alpha_\varepsilon > 0$, depending on i, such that $\varphi_i(\alpha u) < \varepsilon \varphi_i(u)$ for $0 \leq \alpha \leq \alpha_\varepsilon$, $u \geq u_\varepsilon$,

(7) there exist positive constants k, c, u_o and an index i_o such that $\varphi_i(cu) \leq k \varphi_{i_o}(u)$ for $0 \leq u \leq u_o$ and $i \geq i_o$.

$A = (a_{mn\mu\nu})$ denotes a four-dimensional matrix. Throughout this note it will be suppose that (a), (b), and moreover (c), (d) in part. 2.

(a) $a_{mn\mu\nu} \geq 0$ for $m, n, \mu, \nu = 1, 2, \ldots$,

(b) for an arbitrary pair of positive integers (m,n) there exists a pair of positive integers (μ_o, ν_o) such that $a_{mn\mu_o\nu_o} \neq 0$,

(c) $\sup_{m,n} \sum_{\mu,\nu=1}^{\infty} a_{mn\mu\nu} = K < \infty$,

(d) there exists $\lim_{m,n \to \infty} \sum_{\mu,\nu=1}^{\infty} a_{mn\mu\nu}$.

2. Sets of approximable elements. For a given φ-function and a matrix A we define the transforms

$$G_{mn}^{\varphi}(x) = \sum_{\mu,\nu=1}^{\infty} a_{mn\mu\nu} \varphi(|t_{\mu\nu}|)$$

for $m,n = 1, 2, \ldots$.

By a space T_φ^* we understand a set of all sequences $x \in T$ such that $G_{mn}^{\varphi}(\lambda x) < \infty$ for $m,n = 1,2,\ldots$ and $\lim_{m,n \to \infty} G_{mn}^{\varphi}(\lambda x) = 0$ for a certain $\lambda > 0$.

If in this definition we take an arbitrary constant λ or $\lambda = 1$, then the set T_φ^* wil be denoted by T_φ or T_φ^o, respectively. Double sequences $x \in T$ belonging to T_φ^* are called strongly (A, φ)- summable to zero. It is well known that $\varrho_\varphi(x) = \sup_{m,n} G_{mn}^{\varphi}(x)$ for $x \in T_\varphi^o$ and $\varrho_\varphi(x) = \infty$ for $x \in T_\varphi^* \setminus T_\varphi^o$ is a modular, $\|x\|_\varphi = \inf\{\varepsilon > 0 : \varrho_\varphi(\frac{x}{\varepsilon}) \leq \varepsilon\}$ is an F-norm and T_φ^* is semiordered, linear modular space, complete with respect to the norm $\|\cdot\|_\varphi$.

Let (φ_i) be a given sequence of φ-functions and let (ϱ_i) denote a sequence of modulars, where $\varrho_i(x) = \varrho_{\varphi_i}(x)$. By means of this sequence, one may define the following modulars in T :

$$\varrho_W(x) = \sum_{i=1}^{\infty} 2^{-i} \varrho_i(x)(1 + \varrho_i(x))^{-1}, \qquad \varrho_r(x) = \sum_{i=1}^{\infty} 2^{-i} \varrho_i(x),$$

$$\varrho_G(x) = \sup_s \frac{1}{s} \sum_{i=1}^{s} \varrho_i(x), \qquad \varrho_s(x) = \sum_{i=1}^{\infty} \varrho_i(x),$$

$$\varrho_0(x) = \sup_i \varrho_i(x).$$

Let $\widetilde{\varrho}$ be any of the symbols : ϱ_w, ϱ_r, ϱ_σ, ϱ_s and ϱ_o . We define the following sets

$$X_{\widetilde{\varrho}} = \{ x \in T : \widetilde{\varrho}(\lambda x) \longrightarrow 0 \text{ as } \lambda \longrightarrow 0+ \} .$$

Elements $x \in T$ belonging to $X_{\widetilde{\varrho}}$ will be called w, r, σ, s, o-approximable by ϱ_i , respectively. In the following we will write briefly: w-appr, r-appr, σ-appr, s-appr and o-appr , respectively.

3. Elementary properties.

Remark 1. (a) The element $x \in T$ is w-appr, if and only if, $\varrho_i(\lambda x) \longrightarrow 0$ as $\lambda \longrightarrow 0+$ for every i, separately.

(b) The element $x \in T$ is o-appr, if and only if, $\varrho_i(\lambda x) \longrightarrow 0$ as $\lambda \longrightarrow 0+$ uniformly with respect to i.

Remark 2. (a) If $x \in T$ is σ-appr, then is also w-appr.

(b) If $x \in T$ is o-appr, then is σ-appr and w-appr.

(c) Every o-appr or s-appr element is r-appr.

(d) Every s-appr element is o-appr and is also σ-appr and w-appr.

(e) If $x \in T$ is r-appr, then is also w-appr.

Remark 3. If φ-functions φ_i satisfy conditions (1) and (4), then the set of all o-appr elements is identical with the space T_b.

Remark 4. If φ-functions φ_i satisfy conditions (2) and (3), then the space T_b is identical with the set of all s-appr elements.

Remark 5. If φ-functions φ_i satisfy conditions (2) and (4), then sets of all o-appr elements and s-appr elements are identical.

4. Theorems and remarks.

Theorem 1. Let $\liminf_{\mu,\nu \to \infty} a_{m_o n_o \mu\nu} > 0$ for fixed m_o and n_o. If the φ-functions φ_i satisfy conditions (1) and (7), then every w-appr element is also o-appr.

Proof. Let the element $x = (t_{\mu\nu}) \in T$ be w-appr, then there are $\lambda_i > 0$ such that

$$\sup_{m,n} \sum_{\mu,\nu=1}^{\infty} a_{mn\mu\nu} \varphi_i(\lambda_i |t_{\mu\nu}|) < \infty$$

for all i. Hence $x \in T_b$ and moreover $\lim_{\mu,\nu \to \infty}' a_{m_o n_o \mu\nu} \varphi_i(\lambda_i |t_{\mu\nu}|) = 0$ for each i and for fixed m_o and n_o. Thus $\lim_{\mu,\nu \to \infty} \varphi_i(\lambda_i |t_{\mu\nu}|) = 0$ for each i, and by continuity of φ_i , $\lim_{\mu,\nu \to \infty} t_{\mu\nu} = 0$. In consequence we have $x \in T_o \cap T_b$.

From (7) it follows that there are $k, c_o, u_o > 0$ and an index i_o such that $\varphi_i(u) \leq k \varphi_{i_o}(c_o u)$ for $0 \leq u \leq u_o$, $i \geq i_o$. Taking λ sufficiently small, we have $|\lambda t_{\mu\nu}| \leq u_o$ for all μ and ν. Hence $\varphi_i(\lambda |t_{\mu\nu}|) \leq k \varphi_{i_o}(c_o \lambda |t_{\mu\nu}|)$ for $i \geq i_o$ and all μ, ν, i.e. $\varrho_i(\lambda x) \leq k \varrho_{i_o}(c_o \lambda x)$ for $i \geq i_o$. Thus the element x is o-appr.

Remark 6. By Remark 2(b) and Theorem 1 we have that set of all o-appr elements is identical with the set of all w-appr elements.

Theorem 2. Let us suppose that there exist sequences (μ_r), (v_s) such that

(e)
$$\sum_{r,s=1}^{\infty} a_{m_o n_o \mu_r v_s} = \infty$$

for fixed indices m_o, n_o, and $\sup_{m,n} a_{mn\mu_r v_s} \leq L$ for all r, s, where $L > 0$ is a constant.

Let the φ-functions φ_i satisfy (5) and (6). If the set of all w-appr elements is identical with the set of all o-appr elements, then φ_i satisfy (7).

Proof. Let us suppose $X_{\varrho_w} = X_{\varrho_o}$. If (7) does not hold, then for every $k, c, u_o > 0$ and every i_o there exist $0 \leq u \leq u_o$ and $i \geq i_o$ such that $\varphi_i(cu) > k \varphi_{i_o}(u)$. For positiwe integers k, p, q we choose $c = 2^{-k}$ and $i_o = p$, $u_o = q^{-1}$. Then there exist $i_{p,q,k} \geq p$ and $u_{p,q,k} \leq q^{-1}$ such that

(i)
$$\varphi_{i_{p,q,k}}(2^{-k} u_{p,q,k}) > 2^k \varphi_p(u_{p,q,k})$$

for $p, q, k = 1, 2, \ldots$.

Now, we define (as in [5]) increasing sequences of indices (q_k) as follows. We choose q_1 so large that $q_1 \geq \frac{1}{v_1}$, $v_1 > 0$ and $\varphi_1\left(\frac{1}{q_1}\right) \leq \min\{1, (2L)^{-1}\}$, and we put $u_1 = u_{1,q_1,1}$. Let the indices $q_1, \ldots, q_{k-1}$ and the positive numbers $u_1, \ldots, u_{k-1}$ are already chosen in such a manner that $\varphi_i\left(\frac{1}{q_i}\right) \leq 1$, $q_i \geq \frac{1}{v_i}$, $q_1 < q_2 < \ldots < q_{k-1}$ and $\varphi_i\left(\frac{1}{q_i}\right) \leq \varphi_{i-1}(u_{i-1})$, where $u_i = u_{i,q_i,i}$, $i = 2, 3, \ldots, k-1$. Now we take q_k such that $\varphi_k\left(\frac{1}{q_k}\right) \leq \min\{1, 2^{-k} L^{-1}\}$, $q_k \geq \frac{1}{v_k}$ and $\varphi_k\left(\frac{1}{q_k}\right) \leq \varphi_{k-1}(u_{k-1})$ and we put $u_k = u_{k,q_k,k}$. Since $u_k \leq \frac{1}{q_k}$ we have $\varphi_k(u_k) \leq \varphi_k\left(\frac{1}{q_k}\right) \leq \varphi_{k-1}(u_{k-1})$ for $k = 2, 3, \ldots$.

Now we shall construct a sequence (A_k) of pairwise disjoint sets of pairs of indices such that

(ii) $$\frac{1}{2^k} < \sup_{m,n} \sum_{r,s} a_{mn\mu_r\nu_s} \varphi_k(u_k) \leqslant \frac{1}{2^{k-1}}$$

where $A_k = \{(\mu_r, \nu_s)\}$. It is sufficient to construct one set $A_k = \{\mu_{r_1}, \mu_{r_2}, \dots\} \times \{\nu_{s_1}, \nu_{s_2}, \dots\}$ in such a manner that μ_{r_l} , ν_{s_t} are arbitrarily large, respectively. If these sets do not exist then we have three possibilities.

If the inequality

(iii) $$\sup_{m,n} \sum_{l,t=1}^{M,N} a_{mn\mu_{r_l}\nu_{s_t}} \varphi_k(u_k) \leqslant \frac{1}{2^k}$$

holds for all sequences (r_l) and (s_t) and all $m,n \geqslant 1$, then we take for instance $r_l = r_1 + (r-1)$ with respective (s_t) and $M,N \longrightarrow \infty$, and the assumption (e) yields a contradiction.

If there are sequences (r_l), (s_t) and $m,n \geqslant 1$ such that the inequality (iii) holds, but

(iv) $$\sup_{m,n} \sum_{l,t=1}^{M+1,N+1} a_{mn\mu_{r_l}\nu_{s_t}} \varphi_k(u_k) > \frac{1}{2^{k-1}} ,$$

then there exist indices $m=m_k$, $n=n_k$ such that

$$\sum_{l,t=1}^{M,N} a_{m_k n_k \mu_{r_l}\nu_{s_t}} \leqslant \frac{1}{2^k} \quad \text{and} \quad \sum_{l,t=1}^{M+1,N+1} a_{m_k n_k \mu_{r_l}\nu_{s_t}} > \frac{1}{2^{k-1}} .$$

Thus $a_{m_k n_k \mu_{r_{M+1}}\nu_{s_{N+1}}} \varphi_k(u_k) > \frac{1}{2^k}$ and $\frac{1}{2^k} \leqslant a_{m_k n_k \mu_{r_{M+1}}\nu_{s_{N+1}}} \varphi_k(u_k) \leqslant$

$L\varphi_k(u_k) \leqslant L\varphi_k\left(\frac{1}{q_k}\right) \leqslant \frac{1}{2^k}$, a contradiction.

If the inequality (iv) holds always, then the contradiction is obtained as previously, taking for instance $M+1 = r_1$.

Let us take $x = (t_{\mu\nu})$, where

$$t_{\mu\nu} = \begin{cases} u_k & \text{for } (\mu,\nu) \in A_k , \ k=1,2,\dots , \\ 0 & \text{elsewhere.} \end{cases}$$

Evidently, by (ii) and (5) we have

$$\varsigma_i(\lambda_i x) = \sup_{m,n} \sum_{\mu,\nu=1}^{\infty} a_{mn\mu\nu} \varphi_i(\lambda_i |t_{\mu\nu}|) = \sup_{m,n} \sum_{k=1}^{i-1} \sum_{(\mu,\nu)\in A_k} a_{mn\mu\nu} \varphi_i(\lambda_i u_k)$$

$$+ \sup_{m,n} \sum_{k=i}^{\infty} \sum_{(\mu,\nu)\in A_k} a_{mn\mu\nu} \varphi_i(\lambda_i u_k) \leqslant \frac{(i-1)\varphi_i(\lambda_i u_k)}{2^{k-1}\varphi_k(u_k)} + \beta_i \sum_{k=1}^{\infty} 2^{1-k} < \infty .$$

Hence, $\varsigma_i(\lambda_i x) < \infty$.

Let us take in (6) , $\varepsilon = \dfrac{c}{2\varsigma_i(\lambda_i x)}$, and let $\lambda_i' > 0$ be so small that $\lambda_i' \leqslant \lambda_i$ and $\lambda_i' u_k \leqslant u_\varepsilon$ for all k. Moreover, let $\varepsilon' = \dfrac{c}{2\varsigma_i(\lambda_i' x)} \geqslant \varepsilon$.

Hence $u_{\varepsilon'} \geqslant u_\varepsilon$ and so $\lambda_i' u_k \leqslant u_{\varepsilon'}$. Finally

$$\varsigma_i(\lambda x) \leqslant \varepsilon' \sup_{m,n} \sum_{k=1}^{\infty} \sum_{(\mu,\nu)\in A_k} a_{mn\mu\nu} \varphi_i(\lambda_i' |u_k|) = \varepsilon' \varsigma_i(\lambda_i' x) = \tfrac{1}{2}c ,$$

and so x is w-appr.

Now, by the inequalities (i) and (ii) we have

$$\varrho_{i_{k},q_{k},k}(2^{-k}x) \geq \sup_{m,n} \sum_{(\mu,\nu)\in A_k} a_{mn\mu\nu}\varphi_{i_{k},q_{k},k}(2^{-k}u_k) \geq$$

$$\geq 2^k \varphi_k(u_k) \sup_{m,n} \sum_{(\mu,\nu)\in A_k} a_{mn\mu\nu} \geq 1 ,$$

and in consequence the element x is not o-appr. Finally, the sets of all elements w-appr and o-appr are not identical, a contradiction.

Theorem 3. Let us suppose that there exist sequences of indices (μ_r) and (ν_s) such that $\lim_{r,s\to\infty} a_{m_o n_o \mu_r \nu_s} > 0$ for fixed indices m_o, n_o, and $\sup_{m,n} a_{mn\mu_r\nu_s} \leq L$ for all r,s , where $L > 0$ is a constant.

Let the φ-functions φ_i satisfy conditions (1), (5) and (6). Then the sets of all elements w-appr and o-appr are identical, if and only if, φ_i satisfy (7).

Remark 7. Under the same assumptions as in Theorem 1, every w-appr element is also σ-appr.

Remark 8. Applying Remark 4 and condition (1) we have
(a) if every s-appr element is bounded, then identity of the sets of all elements o-appr and s-appr implies the condition (4),
(b) if every bounded element is s-appr, then the condition (4) implies that the sets of all elements o-appr and s-appr are identical.

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