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THE SUMMATION OF BIORTHOGONAL EXPANSION OF RATIONAL FUNCTION IN THE COMPLEX PLANE

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Let G be a simply connected domain, the Boundary of which is a rectifiable Jordan curve Γ and D be a complement of the $\overline{G}$. We denote the function, mapping D conformally onto $|w|>1$ by $w=\phi(z)$, $\phi(\infty)=\infty, \phi'(\infty)>0$.

Definition [1] The domain G is said to belong to K_r, $r>1$, if for any function $g(\zeta)\in L^r(\Gamma)$, the integral

$$G_+(z)=\frac{1}{2\pi i}\int_\Gamma \frac{g(\zeta)}{\zeta-z}\,d\zeta, \qquad z\in G$$

determines a function $G_+(z)\in E^r(G)$ and

$$\|G_+\|_{L^r(\Gamma)} \le C\,\|g\|_{L^r(\Gamma)} ,$$

where C is a constant

Hereafter we denote by C constant of different values.

It is known [2] if $\Gamma\in BAC$, it means that the shorter are length $\widehat{\zeta_1\zeta_2}$ between two points ζ_1 and ζ_2 on Γ satisfies

$$\frac{\widehat{\zeta_1\zeta_2}}{|\zeta_1-\zeta_2|}\le C ,$$

then the domain G whose boundary is Γ is belonging to K_r domain, where $r > 1$ is a arbitrary number.

Hereafter we always assume the domain G is K_r domain, $r > 1$ or $\Gamma \subset BAC$.

Suppose the sequence of different numbers $\{\lambda_k\}$ is belonging to D. Consider the system

$$r_{ks}(z) = \frac{s!}{(z-\lambda_k)^{s+1}}, \quad k=1,2,\cdots, \ s=0, 1,\cdots, m_k-1; \tag{1}$$

where m_k is a number depending on k, $k=1,2,\cdots$.

It is well known [1] the necessary and sufficient condition for the completeness of $\{r_{ks}(z)\}$ in the space $E^p(G)$, $p > 1$ is

$$\sum_{k=1}^{+\infty} m_k\left(1-\frac{1}{|\alpha_k|}\right) = +\infty, \quad \alpha_k = \phi(\lambda_k), \quad k=1,2,\cdots \tag{2}$$

Hence if

$$\sum_{k=1}^{+\infty} m_k\left(1-\frac{1}{|\alpha_k|}\right) < +\infty, \tag{3}$$

then the closure $R_p(G, \lambda_k)$ of $\{r_{ks}(z)\}$ is a real subspace of $E^p(G)$, $p > 1$.

In this paper the summation of biorthogonal expansion of $\{r_{ks}(z)\}$ of the space $E^p(G)$ in the domain G is discussed.

Under the condition (3) consider the system

$$\Omega_{ksn}(z) = \frac{B_n(\phi(z))}{s!\ 2\pi i}\int_{C_k} \frac{(\zeta-\lambda_k)^s}{B_n(\phi(\zeta))}\,\frac{d\zeta}{\zeta-z}, \quad z \in D, \tag{4}$$

$$k=1,2,\cdots, n, \qquad s=0,1,\cdots, m_k-1.$$

and

$$\Omega_{ks}(z) = \frac{B(\phi(z))}{s!\ 2\pi i}\int_{C_k} \frac{(\zeta-\lambda_k)^s}{B(\phi(\zeta))}\cdot\frac{d\zeta}{\zeta-z}, \quad z \in D, \tag{5}$$

$$k=1,2,\cdots, \quad s=0, 1, \cdots, m_k-1,$$

where C_k is a small circle including the λ_k and not including the other $\lambda_j \neq \lambda_k$ and z, and

$$B(w) = \prod_{k=1}^{+\infty} \frac{w - \alpha_k}{1 - \bar{\alpha}_k w} \frac{|\alpha_k|}{\alpha_k}, \quad \alpha_k = \phi(\lambda_k) \tag{6}$$

$$B_n(w) = \prod_{k=1}^{n} \frac{w - \alpha_k}{1 - \bar{\alpha}_k w} \frac{|\alpha_k|}{\alpha_k}, \quad \alpha_k = \phi(\lambda_k) . \tag{7}$$

It is easy to prove

<u>Theorem 1</u>

$$\int_\Gamma \Omega_{ksn}(z) r_{pq}(z) dz = \begin{cases} 1, & k=p \leq n, \quad q=0,1,\cdots, m_p-1 \\ & \qquad s=0,1,\cdots, m_k-1 , \\ 0, & \text{in other cases.} \end{cases} \tag{8}$$

$$\int_\Gamma \Omega_{ks}(z) r_{pq}(z) dz = \begin{cases} 1, & k=p, \quad q=0,1,\cdots, m_p-1 \\ & \qquad s=0,1,\cdots, m_k-1 , \\ 0, & \text{in other cases.} \end{cases} \tag{9}$$

If all $m_k=1$, $k=1,2,\cdots$, then

$$\Omega_{kon}(z) = \frac{-B_n(\phi(z))}{(z-\lambda_k)(B_n(\phi(\lambda_k)))'} , \qquad k=1,2,\cdots .$$

$$\Omega_{ko}(z) = \frac{-B(\phi(z))}{(z-\lambda_k)(B(\phi(\lambda_k)))'} , \qquad k=1,2,\cdots .$$

it is the basic functions in Lagrange interpolation problem.

We say that $f(z) \in E^p(G; \lambda_k)$, $p > 1$, if

1° $f(z) \in E^p(G)$;

2° $f(\zeta)B(\phi(\zeta))$ is the boundary value of some function

$g(z) \in E_0^p(D)$, it means $g(z) \in E^p(D)$ and $g(\infty) = 0$.

Theorem 2 Under the condition (3)

$$R_p(G;\ \lambda_k) = E^p(G;\ \lambda_k), \quad p > 1 \quad . \tag{10}$$

It can easily be proved using the Hahn-Banach theorem and the representation of linear functional in $E^p(G)$, $p > 1$.

Theorem 3 Let $f \in E^p(G)$, $p > 1$, then

$$f(z) = \lim_{n \to +\infty} \sum_{k=1}^{n} \sum_{s=0}^{m_k-1} a_{ksn}(f) r_{ks}(z)$$

$$+ \frac{1}{2\pi i} \int_\Gamma f(\tilde{\zeta}) B(\phi(\tilde{\zeta}))d\tilde{\zeta} \cdot \frac{1}{2\pi i} \int_\Gamma \frac{dt}{B(\phi(t))(t-\tilde{\zeta})(z-t)},$$

$$z \in G \tag{11}$$

where

$$a_{ksn}(f) = \int_\Gamma f(\zeta)\Omega_{ksn}(\zeta)d\zeta \quad , \tag{12}$$

the "limit" is understood as convergence in $E^p(G)$ and $\tilde{\zeta}$ is the boundary value when $\zeta \in D$ tends to Γ.

Proof Consider the projection of $f(z)$ on the linear space E_n constructed by $\{r_{ks}(z)\}$, $k=1,2,\cdots, n$, $s=0,1,\cdots, m_k-1$:

$$(P_{E_n} f)(z) = \sum_{k=1}^{n} \sum_{s=0}^{m_k-1} a_{ksn}(f) r_{ks}(z)$$

$$= \int_\Gamma f(\zeta)K(z,\ \zeta)d\zeta \ , \tag{13}$$

where

$$K(z,\ \zeta) = \sum_{k=1}^{n} \sum_{s=0}^{m_k-1} r_{ks}(z)\ \Omega_{ksn}(\zeta)$$

$$= \sum_{k=1}^{n} \sum_{s=0}^{m_k-1} \frac{B_n(\phi(\zeta))}{(2\pi i)^2} \int_{c_k} \frac{dt}{B_n(\phi(t))(t-\zeta)(z-\lambda_k)} \left(\frac{t-\lambda_k}{z-\lambda_k}\right)^s .$$

Since the function

$$\sum_{s=m_k}^{+\infty} \left(\frac{t-\lambda_k}{z-\lambda_k}\right)^s / B_n(\phi(z))$$

is analytic at $t=\lambda_k$, it follows from the Cauchy theorem

$$K(z, \zeta) = \sum_{k=1}^{n} \frac{B_n(\phi(\zeta))}{(2\pi i)^2} \int_{c_k} \frac{1}{B_n(\phi(t))(t-\zeta)(z-\lambda_k)} \sum_{s=0}^{+\infty} \left(\frac{t-\lambda_k}{z-\lambda_k}\right)^s dt$$

$$= \sum_{k=1}^{n} \frac{B_n(\phi(\zeta))}{(2\pi i)^2} \int_{c_k} \frac{dt}{B_n(\phi(t))(t-\zeta)(z-t)}$$

$$= \frac{B_n(\phi(\zeta))}{2\pi i} \left[\frac{1}{B_n(\phi(\zeta))(\zeta-z)} - \frac{1}{2\pi i} \int_{\Gamma} \frac{dt}{B_n(\phi(t))(t-\zeta)(z-t)}\right]$$

$$= \frac{1}{2\pi i} \frac{1}{\zeta-z} - \frac{B_n(\phi(\zeta))}{(2\pi i)^2} \int_{\Gamma} \frac{dt}{B_n(\phi(t))(t-\zeta)(z-t)} ,$$

$$\zeta \in D, \quad z \in G. \qquad (14)$$

Let $\zeta \to \tilde{\zeta} \in \Gamma$, from (14) and (13) we obtain

$$(P_{E_n} f)(z) = f(z) - \frac{1}{2\pi i} \int_{\Gamma} f(\tilde{\zeta}) B_n(\phi(\tilde{\zeta})) d\zeta \cdot \frac{1}{2\pi i} \int_{\Gamma} \frac{dt}{B_n(\phi(t))(t-\tilde{\zeta})(z-t)} .$$

(15)

We will prove that

$$I_n = \frac{1}{2\pi i} \int_{\Gamma} f(\tilde{\zeta}) B_n(\phi(\tilde{\zeta})) d\tilde{\zeta} \cdot \frac{1}{2\pi i} \int_{\Gamma} \frac{dt}{B_n(\phi(t))(t-\tilde{\zeta})(z-t)} .$$

Converges to

$$I = \frac{1}{2\pi i} \int_\Gamma f(\tilde{\zeta}) B_n(\phi(\tilde{\zeta}))\, d\tilde{\zeta} \cdot \frac{1}{2\pi i} \int_\Gamma \frac{dt}{B(\phi(t)(t-\tilde{\zeta})(z-t)} \tag{16}$$

is the space $L^p(\Gamma)$.

In fact we have

$$I_n = \frac{1}{2\pi i} \text{ V.P. } \int_\Gamma \frac{f(\zeta) B_n(\phi(\zeta))\, d\zeta}{t-\zeta} \cdot \frac{1}{2\pi i} \int_\Gamma \frac{dt}{B_n(\phi(t))(z-t)}$$

$$- \frac{1}{2} \frac{1}{2\pi i} \int_\Gamma \frac{f(\zeta)}{z-\zeta}\, d\zeta$$

$$= \frac{1}{2\pi i} \text{ V.P. } \int_\Gamma \frac{f(\zeta) B_n(\phi(\zeta))\, d\zeta}{t-\zeta} \cdot \frac{1}{2\pi i} \int_\Gamma \frac{dt}{B_n(\phi(t))(z-t)} + \frac{1}{2} f(z).$$

and

$$I = \frac{1}{2\pi i} \text{ V.P. } \int_\Gamma \frac{f(\zeta) B(\phi(\zeta))\, d\zeta}{t-\zeta} \frac{1}{2\pi i} \int_\Gamma \frac{dt}{B(\phi(t))(z-t)} + \frac{1}{2} f(z). \tag{17}$$

Using the properties of K_r domain, we get

$$\| I_n - I \|_{L^p(\Gamma)}$$

$$\leq \Big\| \frac{1}{2\pi i} \text{ V.P. } \int_\Gamma \frac{f(\zeta) B_n(\phi(\zeta))\, d\zeta}{t-\zeta} \cdot \frac{1}{2\pi i} \int_\Gamma \frac{dt}{B_n(\phi(t))(z-t)}$$

$$- \frac{1}{2\pi i} \text{ V.P. } \int_\Gamma \frac{f(\zeta) B(\phi(\zeta))}{t-\zeta}\, d\zeta \cdot \frac{1}{2\pi i} \int_\Gamma \frac{dt}{B_n(\phi(t))(z-t)} \Big\|_{L^p(\Gamma)}$$

$$+ \Big\| \frac{1}{2\pi i} \text{ V.P. } \int_\Gamma \frac{f(\zeta) B(\phi(\zeta))}{t-\zeta}\, d\zeta \frac{1}{2\pi i} \int_\Gamma \frac{dt}{B_n(\phi(t))(z-t)}$$

$$- \frac{1}{2\pi i} \text{ V.P. } \int_\Gamma \frac{f(\zeta) B(\phi(\zeta))}{t-\zeta}\, d\zeta \frac{1}{2\pi i} \int_\Gamma \frac{dt}{B(\phi(t))(z-t)} \Big\|_{L^p(\Gamma)}$$

$$\leq C_1 \left\| \text{V.P.} \int_\Gamma \frac{1}{|B_n(\phi(t))|} \frac{f(\zeta)(B_n(\phi(\zeta))-B(\phi(\zeta)))}{t-\zeta} d\zeta \right\|_{L^p(\Gamma)}$$

$$+ C_2 \left\| \text{V.P.} \int_\Gamma \frac{|B_n(\phi(t))-B(\phi(t))|}{|B_n(\phi(t))|\,|B(\phi(t))|} \cdot \frac{f(\zeta)B(\phi(\zeta))}{t-\zeta} d\zeta \right\|_{L^p(\Gamma)}$$

$$\leq C_3 \left\| f(\zeta)(B_n(\phi(\zeta))-B(\phi(\zeta))) \right\|_{L^p(\Gamma)} + C_4 \| f(\zeta) \|_{L^p(\Gamma)} \cdot$$

$$\| B_n(\phi(t))-B(\phi(t)) \|_{L^p(\Gamma)} .$$

Using the method of [2] if can be proved that the right-hand of side of above inequality tends to zero. Hence we have

$$\| I_n - I \| \to 0 .$$

theorem 3 is proved completely.

<u>Theorem 4</u> The necessary and sufficient condition for $f \in E^p(G, \lambda_k)$ is

$$\frac{1}{2\pi i} \int_\Gamma f(\tilde{\zeta})B(\phi(\tilde{\zeta}))d\tilde{\zeta} \; \frac{1}{2\pi i} \int_\Gamma \frac{dt}{B(\phi(t))(t-\tilde{\zeta})(z-t)} \equiv 0, \quad z \in G.$$

<u>Proof</u> Necessity. From $f \in E^p(G, \lambda_k)$ it follows that $f(\tilde{\zeta})B(\phi(\tilde{\zeta}))$ is the boundary value of some $g(\zeta) \in E_0^p(D)$ on Γ. Hence using (17) we have

$$I = \frac{1}{2\pi i} \text{V.P.} \int_\Gamma \frac{g(\zeta)d\zeta}{t-\zeta} \; \frac{1}{2\pi i} \int_\Gamma \frac{dt}{B(\phi(t))(z-t)} + \frac{1}{2} f(z)$$

$$= \frac{1}{2} \frac{1}{2\pi i} \int_\Gamma \frac{g(t)dt}{B(\phi(t))(z-t)} + \frac{1}{2} f(z)$$

$$= \frac{1}{2} \frac{1}{2\pi i} \int_{\Gamma} \frac{f(t)}{z-t} dt + \frac{1}{2} f(z) \equiv 0, \quad z \in G.$$

Sufficiency. Let

$$g_1(t) = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(\zeta)B(\phi(\zeta))}{\zeta - t} d\zeta \in E_0^p(D).$$

Under the condition of theorem 4 from (17) we have

$$I = \frac{1}{2\pi i} \int_{\Gamma} \frac{-g_1(t) - \frac{1}{2} f(t)B(\phi(t))}{B(\phi(t))(z-t)} dt + \frac{1}{2} f(z) \equiv 0, \quad z \in G,$$

i.e.

$$\frac{1}{2\pi i} \int_{\Gamma} \frac{f(t) - \dfrac{g_1(t)}{B(\phi(t))}}{t - z} dt \equiv 0, \quad z \in G.$$

It follows that there exists some $g_2(\zeta) \in E_0^p(D)$ such that on Γ

$$f(t) - \frac{g_1(t)}{B(\phi(t))} = g_2(t), \quad t \in \Gamma. \text{ a.e.}$$

i.e.

$$f(t)B(\phi(t)) = g_1(t) + g_2(t)B(\phi(t)) \in E_0^p(D).$$

This completes the proof of theorem 4.

<u>Corollary</u> From theorems 3 and 4 it follows that the system $\{r_{ks}(z)\}$ is complete in the space $E^p(G; \lambda_k)$.

Now we are going to investigate the problem on the summation of biorthogonal expansion of $\{r_{ks}(z)\}$ of function in $E^p(G)$ in its closure $R_p(G; \lambda_k)$.

<u>Theorem 5</u> The projection $(p_E f)(z)$ of each function $f(z) \in E^p(G)$, $p > 1$ on $R_p(G; \lambda_k)$ of the incomplete system $\{r_{ks}(z)\}$ has the following representation:

$$(P_E f)(z) = \lim_{n \to +\infty} \sum_{k=1}^{n} \sum_{s=0}^{m_k - 1} a_{ks}(f) \frac{1}{2\pi i} \cdot \frac{d^s}{dt^s}\left(\frac{r_n(\phi(t))}{z-t}\right)\Bigg|_{t=\lambda_k} , \tag{18}$$

$$r_n(w) = \prod_{k=n+1}^{+\infty} \frac{w-\alpha_k}{1-\bar{\alpha}_k w} \; \frac{|\alpha_k|}{\alpha_k} , \quad \alpha_k = \phi(\lambda_k). \tag{19}$$

and

$$a_{ks}(f) = \int_\Gamma f(\zeta)\Omega_{ks}(\zeta)d\zeta , \tag{20}$$

where the "linit" is understood in space $L^p(\Gamma)$, $p > 1$.

Proof Consider the function

$$S_n(z) = \sum_{k=1}^{n} \frac{1}{2\pi i} \int_\Gamma f(\zeta)B(\phi(\zeta))d\zeta \; \frac{1}{2\pi i} \int_{C_k} \frac{dt}{B_n(\phi(t))(t-\zeta)(z-t)}$$

$$= \sum_{k=1}^{n} \frac{1}{2\pi i} \int_\Gamma f(\zeta)B(\phi(\zeta))d\zeta \; \frac{1}{2\pi i} \int_{C_k} \frac{G(t)}{B(\phi(t))(t-\zeta)} dt, \tag{21}$$

where

$$G(t) = \frac{r_n(\phi(t))}{z-t} = \sum_{s=0}^{+\infty} \frac{G^{(s)}(\lambda_k)}{s!} (t-\lambda_k)^s \in E_0^p(D) \tag{22}$$

Using the analyticity of

$$\frac{1}{B(\phi(t))} \sum_{s=m_k}^{+\infty} \frac{G^{(s)}(\lambda_k)}{s!} (t-\lambda_k)^s$$

at $t=\lambda_k$ we have

$$S_n(z) = \sum_{k=1}^{n} \frac{1}{2\pi i} \int_\Gamma f(\zeta)B(\phi(\zeta))d\zeta \; \frac{1}{2\pi i} \int_{C_k} \frac{\sum_{s=0}^{m_k-1} \frac{G^{(s)}(\lambda_k)}{s!} (t-\lambda_k)^s}{B(\phi(t))(t-\zeta)} dt$$

$$= \sum_{k=1}^{n} \sum_{s=0}^{m_k-1} \frac{G^{(s)}(\lambda_k)}{2\pi i} \int_\Gamma f(\zeta)\Omega_{ks}(\zeta)d\zeta$$

$$= \sum_{k=1}^{n} \sum_{s=0}^{m_k-1} a_{ks}(f) \frac{1}{2\pi i} \frac{d^s}{dt^s} \left(\frac{r_n(\phi(t))}{z - t} \right)_{t=\lambda_k} \qquad (23)$$

On the other side we proceed as in obtaining (14) and (15) we have

$$s_n(z) = \frac{1}{2\pi i} \int_\Gamma \frac{B(\phi(\zeta))}{B_n(\phi(\zeta))} \frac{f(\zeta)}{\zeta - z} d\zeta$$

$$- \frac{1}{2\pi i} \int_\Gamma f(\zeta)B(\phi(\zeta))d\zeta \frac{1}{2\pi i} \int_\Gamma \frac{dt}{B_n(\phi(t))(t-\zeta)(z-t)} \qquad (24)$$

Using the method of proving the theorem 3 from (24) we can get

$$\left\| s_n(z) - f(z) + \frac{1}{2\pi i} \int_\Gamma f(\zeta)B(\phi(\zeta))d\zeta \frac{1}{2\pi i} \int_\Gamma \frac{dt}{B(\phi(t))(t-\zeta)(z-t)} \right\|_{L^p(\Gamma)} \longrightarrow 0 \quad (25)$$

Hence by using the theorem 4 from (23)-(25) it follows theorem 5.

References

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